NETWORK ANALYSIS AND DESIGN BY PARAMETER PLANE TECHNIQUES.

FLOYD. H. HOLLISTER

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Submitted in partial fulfillment of
the requirements for the degree of

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IN
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ABSTRACT

Parameter plane techniques of analysis and design are applied to variable parameter linear electric networks. Parameter plane theory is extended in several areas.

A theorem permits the constant zeta and omega loci to be obtained in non-parametric form.

The class of network functions to which parameter plane techniques may be applied is broadened by permitting parameters to appear non-linearly in the polynomial coefficients.

Algebraic design methods are presented which permit the solution for parameter values which will simultaneously meet many different types of specifications.

Pole-zero sensitivity formulas are derived from the non-parametric equations of the constant zeta and omega loci.

Equations for curves of constant bandwidth and Q are derived which permit these characteristics to be displayed on the parameter plane.

Examples are presented which demonstrate the application of these techniques to several common networks.

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1. Introduction

Linear active and passive electric networks frequently have one or more variable parameters. In certain instances network elements are intentionally varied so as to modify the characteristics of the network. In other instances the network has no intentionally variable parameters, but due to component tolerances, aging, heating etc., variations do occur. If it is assumed that these variations occur sufficiently slowly so that the network may be considered time invariant, and if it is desired to predict or control the effects of such variations on the characteristics of the network, then the methods to be described can be applied.

Parameter plane techniques were first introduced as a technique for analysis and synthesis of feedback control systems. They were used to investigate the effects upon the roots of the control system characteristic equation as two parameters within the system were varied.

In the early 1950's Mitrovic (1) considered system parameter variations which directly affected two of the coefficients of the system characteristic equation. For example:

$$a_n s^n + a_{n-1} s^{n-1} + \ldots + a_3 s^3 + a_2 s^2 + B_1 s + B_0 = 0 \qquad (1\text{-}1)$$

where: a_i are constant and real

B_0 and B_1 are variable and real

Mitrovic's original work resulted in curves of constant zeta, omega, and sigma which he drew upon the B_0-B_1 plane and which specified the roots of Equation (1-1) for any choice of B_0 and B_1.

In 1964, Siljak (2) extended Mitrovic's work to include cases where variable system parameters, say α and β appeared linearly in one or

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more of the coefficients of the characteristic equation. Specifically Siljak investigated the case where:

$$\sum_{i=0}^{n} a_i s^i = 0 \tag{1-2}$$

where:

$$a_i = b_i \alpha + c_i \beta + d_i$$

and:

b_i, c_i, d_i are constant and real

α, β are variable and real

Siljak, like Mitrovic, obtained curves which were plotted upon the α-β plane and which specified the roots of Equation (1-2) for any choice of α and β. It was Siljak who first referred to such graphs as "parameter planes".

The parameter plane provides information about the roots of the characteristic equation similar to that provided by root loci, but unlike root locus techniques, which investigate the effects of varying one system parameter, parameter plane techniques are inherently best suited for investigating the effects of varying two system parameters.

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2. Basic Parameter Plane Techniques

Parameter Plane techniques evolved from a study of the characteristic equation of control systems and of how the roots of such equations were affected as two parameters of the control system were simultaneously varied. More basically however, the parameter plane is a method for investigating the roots of variable coefficient polynomials, and hence is applicable in a much wider sphere than that of control systems alone. This chapter will review the parameter plane techniques as presented by Siljak (2) and will point out some of the reasons that have limited the scope of applicability of these techniques. Subsequent chapters will present new work which overcomes some of these limitations.

Let us consider a polynomial equation:

$$F(s) = \sum_{k=o}^{n} a_k s^k = 0 \tag{2-1}$$

where the coefficients $a_k (k = 0, 1, \ldots, n)$ are real and s is the complex frequency variable $s = \sigma + j\omega = -\zeta\omega + j\omega\sqrt{1-\zeta^2}$

If one forms powers of s it can be shown (2) that the following relation holds:

$$s^k = \omega^k (T_k(-\zeta) + j \sqrt{1-\zeta^2}\; U_k(-\zeta)) \tag{2-2}$$

where:

$$\begin{aligned} T_k(-\zeta) &= (-1)^k T_k(\zeta) \\ U_k(-\zeta) &= (-1)^{k+1} U_k(\zeta) \end{aligned} \tag{2-3}$$

The U_k and T_k are Chebyshev functions of the first and second kind respectively and are given by the following recursion relations (2):

$$T_{k+1}(\zeta) - 2\zeta T_k(\zeta) + T_{k-1}(\zeta) = 0$$

$$U_{k+1}(\zeta) - 2\zeta U_k(\zeta) + U_{k-1}(\zeta) = 0 \qquad (2\text{-}4)$$

where:

$$T_0(\zeta) = 1, \; T_1(\zeta) = \zeta, \; U_0(\zeta) = 0, \; U_1(\zeta) = 1$$

Tables of these Chebyshev functions are included in Appendix III.

Substituting Equation (2-2) into Equation (2-1) and setting the real and imaginary parts of the resulting equation equal to zero independently we obtain:

$$\sum_{K=0}^{n} a_k \omega^k T_k(-\zeta) = 0 \qquad (2\text{-}5)$$

$$\sum_{K=0}^{n} a_k \omega^k U_k(-\zeta) = 0 \qquad (2\text{-}6)$$

Employing Equations (2-3) and (2-4) we obtain these expressions in terms of the tabulated Chebyshev functions:

$$\sum_{K=0}^{n} (-1)^k a_k \omega^k U_{k-1}(\zeta) = 0 \qquad (2\text{-}7)$$

$$\sum_{K=0}^{n} (-1)^k a_k \omega^k U_k(\zeta) = 0 \qquad (2\text{-}8)$$

Let us consider, as Siljak did, that the coefficients a_k of Equation (2-1) are linear functions of the variable parameters α and β as follows:

$$a_k = b_k \alpha + c_k \beta + d_k \qquad (2\text{-}9)$$

where:

b_k, c_k, d_k ($k = 0, 1, \ldots, n$) are constant and real

α, β are variable and real

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Substituting Equation (2-9) into Equations (2-7) and (2-8) we obtain:

$$B_1\alpha + C_1\beta = -D_1 \qquad (2\text{-}10)$$
$$B_2\alpha + C_2\beta = -D_2$$

where:

$$B_1 = \sum_{K=0}^{n} (-1)^k b_k \omega^k U_{k-1} \qquad B_2 = \sum_{K=0}^{n} (-1)^k b_k \omega^k U_k$$
$$C_1 = \sum_{K=0}^{n} (-1)^k c_k \omega^k U_{k-1} \qquad C_2 = \sum_{K=0}^{n} (-1)^k c_k \omega^k U_k \qquad (2\text{-}11)$$
$$D_1 = \sum_{K=0}^{n} (-1)^k d_k \omega^k U_{k-1} \qquad D_2 = \sum_{K=0}^{n} (-1)^k d_k \omega^k U_k$$

Equations (2-10) are two linear equations in the variables α and β. Solving these equations we obtain:

$$\alpha = \frac{C_1D_2 - C_2D_1}{B_1C_2 - B_2C_1} \qquad \beta = \frac{B_2D_1 - B_1D_2}{B_1C_2 - B_2C_1} \qquad (2\text{-}12)$$

Equations (2-12) give α and β in terms of ζ and ω. Thus if we fix $\zeta = \zeta_0$ and vary ω over the range $0 < \omega < \infty$ we will obtain a curve in the α - β plane which specifies the relation between α and β necessary to cause a pair of complex roots of Equation (2-1) to have a damping ratio ζ equal to ζ_0. Curves resulting from fixed values of ζ are referred to as constant zeta loci. If we fix $\omega = \omega_0$ and vary ζ over the range $-1 < \zeta < +1$ we will obtain another curve in the α - β plane which specifies the relation between α and β necessary to cause a pair of complex roots of Equation (2-1) to have an undamped natural frequency $\omega = \omega_0$. Curves drawn for fixed values of ω are referred to as constant omega curves.

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If we substitute $s = \sigma$ in Equation (2-1), where sigma refers to values of s along the real axis in the s-plane, then we obtain:

$$\sum_{K=0}^{n} a_k \sigma^k = 0 \qquad (2\text{-}13)$$

If we now substitute for a_k the linear relation (2-9) we obtain:

$$\alpha \sum_{K=0}^{n} b_k \sigma^k + \beta \sum_{K=0}^{n} c_k \sigma^k + \sum_{K=0}^{n} d_k \sigma^k = 0 \qquad (2\text{-}14)$$

For fixed values of sigma, Equation (2-13) represents a straight line in the $\alpha - \beta$ plane. This straight line specifies the relation between α and β necessary for Equation (2-1) to have a real root at $s = \sigma$. Such lines drawn upon the $\alpha - \beta$ plane are referred to as constant sigma loci.

If the reader is interested in the mapping which occurs between the s-plane and the parameter plane or in the properties of this mapping then the author refers him to reference (2) where this is discussed in detail.

Example I:

Let us consider that we have constructed a parameter plane for some third order polynomial and that this appears as shown in Figure 1.

Let us consider that parameter values α and β are chosen to be α_0 and β_0 respectively, and that at these coordinates on the parameter plane it happens that the σ_1, ζ_1 and ω_2 loci intersect. We may thus assert that for this choice of parameter values the polynomial has the roots:

$$s = \sigma_1$$

$$s = -\zeta_1 \omega_2 \pm j\,\omega_2(1-\zeta_1^{\,2})^{\frac{1}{2}}$$

Let us now presume that we wish to change the damping ratio to ζ_0,

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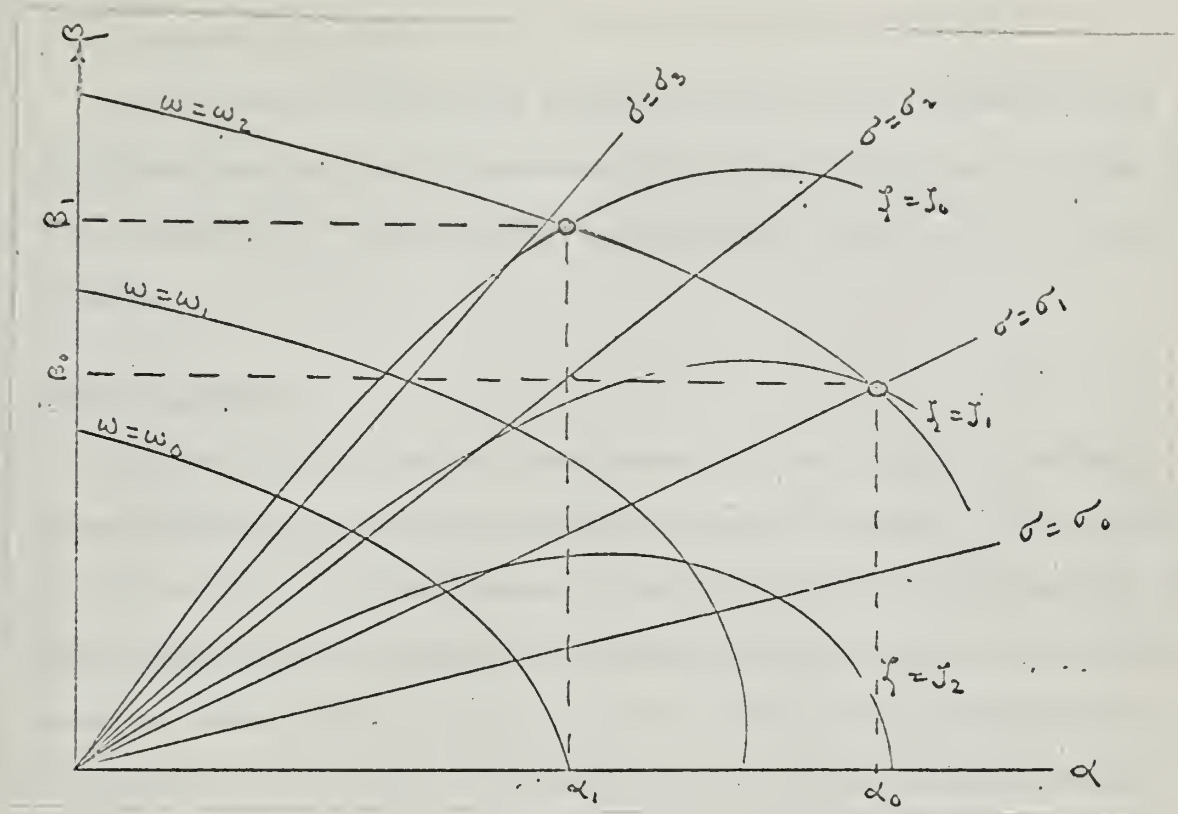

Fig. 1. Example of a Parameter for some 3rd Order Polynomial

but wish to leave the undamped natural frequency unchanged at ω_2. We therefore find the intersection of the ζ_0 and ω_2 loci. In this example this occurs at $\alpha = \alpha_1$, $\beta = \beta_1$. Adjusting the parameters α and β to these values will guarantee complex roots at $s = -\zeta_0 \omega_2 \pm j\,\omega_2 (1-\zeta_0^2)^{\frac{1}{2}}$. The real root which was at $s = \sigma_1$ has moved to a new location between σ_2 and σ_3.

Thus we see that the parameter plane is a general tool of both analysis and synthesis in two parameter systems.

3. Scope of this Work

This section outlines the general content of this manuscript and introduces new work which overcomes some of the limitations of parameter plane theory which have heretofore prevented its application to electric networks.

General Content

In this work the author investigates the application of parameter plane techniques to variable parameter electric networks. The objective of this work is to provide means of analysis which will show how the network characteristics change as the parameters are varied and to provide means of design which will permit values of the adjustable parameters to be chosen so as to cause the network to have desired characteristics. To this end both graphical and analytic techniques are presented. The network characteristics which can either be determined or controlled are pole and zero locations, frequency response, Q and root sensitivity.

Section 4 through 8 develope the mathematical theory which related the network characteristics to the variable parameters. The mathematical theory presented in these sections forms the basis for both the graphical and analytic techniques.

Sections 9, 10, and 11 contain examples of graphical analysis and design of selected networks which have historically been of considerable interest. These examples provide not only an illustration of the potential of the parameter plane techniques, but also provide normalized results which will aid designers in employing these networks.

Section 12 considers several other networks of general interest. These networks are considered in enough detail to show the applicability

of parameter plane techniques to their analysis and design.

Section 13 presents an algebraic method of design which employs the parameter plane equations developed in earlier sections. These algebraic methods of design can be applied to any problem which may be graphically designed and additionally to the much larger class of problem which have more than two parameters. There is theoretically no limitation on the number of parameters which may be simultaneously considered when using these algebraic techniques of design.

Section 14 concludes the manuscript with the author's comments on the work and with his recommendations for additional work in the area of network analysis and design by parameter plane techniques.

Limitations of Earlier Parameter Plane Theory and Introduction to New Work

Parameter plane theory as it has heretofore existed has several limitations which have restricted its application to electric network analysis and design. The remainder of this section discusses these limitations and introduces new work aimed at overcoming them.

Form of the Coefficients

Consider the restrictions which have been placed on the polynomial coefficients - namely that the variable parameters must appear linearly in the coefficients:

$$a_k = b_k \alpha + c_k \beta + d_k$$

This restriction was not too severe when parameter plane techniques were applied primarily to feedback control systems since many useful systems have characteristic equation coefficients of this form. Electric networks on the other hand, seldom have coefficients of this type. Consider for

example the Parallel-T network shown in Figure 2:

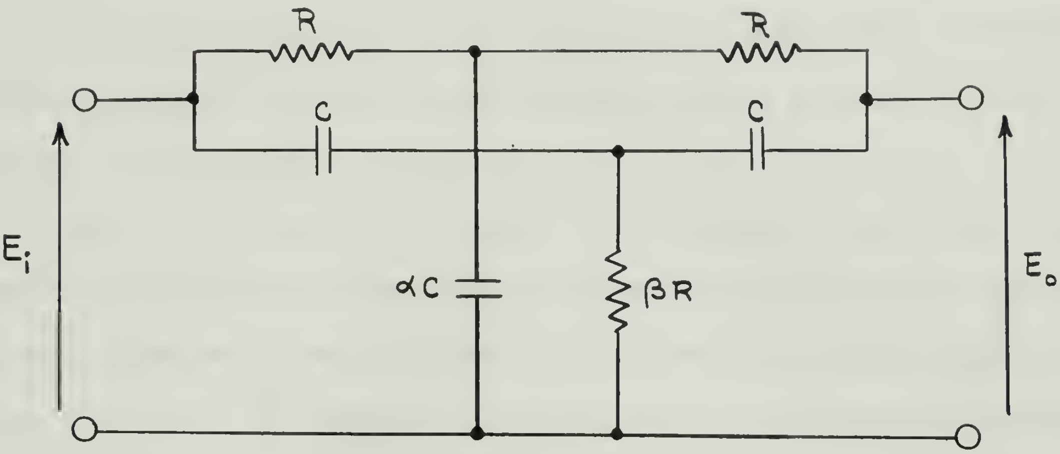

Fig. 2. Symmetrical Parallel-Tee Network with Variable Elements

This network has the transfer function:

$$\frac{E_o(p)}{E_i(p)} = \frac{\alpha\beta p^3 + 2\beta p^2 + 2\beta p + 1}{\alpha\beta p^3 + (2\beta + 2\alpha\beta + \alpha)p^2 + (2\beta + \alpha + 2) + 1} \qquad (3\text{-}2)$$

where: $p = RCs$

Observe that Equation (3-2) contains terms $\alpha\beta$ as well as the linear terms permitted by Equation (3-1). Thus we are unable at this point, due to the linear requirements placed on the coefficients, to determine by parameter plane techniques how the singularities of (3-2) are altered as the parameters α and β are varied. New work to be presented in Section 4 will overcome this difficulty.

Constant Zeta and Omega Loci

As derived by Siljak (2), the constant zeta and omega loci are presented in parametric form as functions of the independent variables omega and zeta respectively. This may be seen in Equation (2-12) where

the constant zeta loci are obtained by fixing zeta at selected values and generating the loci by varying the independent variable omega over the range 0 to infinity. Similarily, the constant omega loci are obtained by fixing omega at selected values and then varying zeta as an independent variable over the range -1 to +1.

Clearly it is desirable to obtain the equations of these loci in non-parametric form. By achieving this we simplify the plotting of these loci, and more important we obtain these loci as families of plane algebraic curves in the α - β plane. This will permit the mathematical procedures for the investigation of plane algebraic curves to be applied when analyzing the behavior of these loci. This is an important step if procedures for sketching the constant zeta and omega loci are to be developed. In Section 5, procedures are presented for reducing these equations to non-parametric form.

Frequency Response Characteristics

Since frequency response characteristics are often used in specifying the behavior of an electric network it is essential that any method which purports to be a general tool for analysis and design of such a network include procedures for determining and controlling the essential features of its frequency response. Choe (3) developed formulas for displaying curves of constant bandwidth upon the Mitrovic plane, but these techniques were not applicable to the parameter plane. In Section 6 formulas are developed for displaying curves of constant bandwidth upon the parameter plane. These formulas are not only applicable to electric networks, but to any rational system function which can be displayed upon the parameter plane.

Constant Q Loci

Second order network transfer functions can be characterized by their Q, much in the same way that the characteristic equation of a second order servomechanism is characterized by its damping ratio. Procedures are developed in Section 7 for displaying loci of constant Q upon the parameter plane. These specify the value of Q which results for any choice of network parameters or alternately specify in what fashion the network parameters must be changed in order to obtain some desired value of Q.

Root Sensitivity to Parameter Variations

The sensitivity of the roots of a polynomial to changes in parameters contained in its coefficients can be catagorized as macroscopic or microscopic. Macroscopic sensitivity refers to the sensitivity of the roots to large changes in the parameters. Microscopic sensitivity refers to infinitesmal changes in the parameters. The constant zeta and omega loci drawn upon the parameter plane permit macroscopic sensitivity to be determined by inspection. Microscopic sensitivity may be determined from formulas derived by Kokotovic and Siljak (5) from the parameter plane equations or may be computed directly from the non-parametric equations of the constant zeta and omega loci which are developed in Section 5. Section 8 describes how both macroscopic and microscopic root sensitivity to parameter variations may be determined.

Examples of Parameter Plane Techniques Applied to Networks

Sections 9 - 11 contain detailed examples of parameter plane techniques applied to specific networks. These examples were carefully chosen in order to both fully demonstrate the current state of parameter plane techniques, and at the same time to produce results useful to persons desiring to employ these networks. The results which are presented are general and

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are presented in normalized form.

Section 12 contains examples of several networks with variable parameters which were studied only so far as to show the applicability of parameter plane techniques to their analysis and design.

4. Parameter Plane Equations for Coefficients of the Form:

$$a_k = b_k\alpha + c_k\beta + d_k\alpha\beta + f_k$$

In this section equations will be obtained which will permit parameter planes to be constructed for polynomials whose coefficients are of the form:

$$a_k = b_k\alpha + c_k\beta + d_k\alpha\beta + f_k \tag{4-1}$$

where: b_k, c_k, d_k are constant and real

α, β are variable and real

Consider the polynomial equation:

$$F(s) = \sum_{K=0}^{n} a_k s^k = 0 \tag{4-2}$$

Let the coefficients a_k be of the form specified by Equation (4-1).

Let:

$$s^k = \omega^k[T_k(-\zeta) \pm j\sqrt{1-\zeta^2}\; U_k(-\zeta)] \tag{4-3}$$

where: T_k and U_k are Chebyshev functions discussed in Section 2.

Substituting (4-3) into (4-2) we obtain:

$$F(s) = \sum_{K=0}^{n} a_k\omega^k T_k(-\zeta) \pm j\sqrt{1-\zeta^2} \sum_{K=0}^{n} a_k\omega^k U_k(-\zeta) \tag{4-4}$$

Equating the real and imaginary parts of Equation (4-4) to zero independently we obtain:

$$\sum_{K=0}^{n} a_k\omega^k T_k(-\zeta) = 0 \tag{4-5}$$

$$\sum_{K=0}^{n} a_k\omega^k U_k(-\zeta) = 0 \tag{4-6}$$

but from Equations (2-3) and (2-4):

$$T_k(-\zeta) = (-1)^k\, T_k(\zeta)$$

$$U_k(-\zeta) = (-1)^{k+1}\, U_k(\zeta) \tag{4-7}$$

$$U_{k-1}(-\zeta) = (-1)^k\, U_{k-1}(\zeta)$$

Substituting Equations (4-7) into Equations (4-5) and (4-6) we obtain:

$$\sum_{K=0}^{n} (-1)^k a_k\, \omega^k\, T_k(\zeta) = 0 \tag{4-8}$$

$$\sum_{K=0}^{n} (-1)^k a_k\, \omega^k\, U_k(\zeta) = 0 \tag{4-9}$$

Substituting for the coefficients a_k in Equations (4-8) and (4-9) the form specified by (4-1) we obtain:

$$\begin{aligned} &\alpha \sum_{K=0}^{n} (-1)^k b_k\, \omega^k T_k(\zeta) + \beta \sum_{K=0}^{n} (-1)^k c_k\, \omega^k T_k(\zeta) + \\ &\alpha\beta \sum_{K=0}^{n} (-1)^k d_k\, \omega^k T_k(\zeta) + \sum_{K=0}^{n} (-1)^k f_k\, \omega^k T_k(\zeta) = 0 \end{aligned} \tag{4-10}$$

$$\begin{aligned} &\alpha \sum (-1)^k b_k\, \omega^k U_k(\zeta) + \beta \sum_{K=0}^{n} (-1)^k c_k\, \omega^k U_k(\zeta) + \\ &\alpha\beta \sum (-1)^k d_k\, \omega^k U_k(\zeta) + \sum_{K=0}^{n} (-1)^k f_k\, \omega^k U_k(\zeta) = 0 \end{aligned} \tag{4-11}$$

which may be written:

$$B_1\alpha + C_1\beta + D_1\alpha\beta + F_1 = 0 \tag{4-12}$$

$$B_2\alpha + C_2\beta + D_2\alpha\beta + F_2 = 0 \tag{4-13}$$

where:

$$\begin{aligned} B_1 &= \sum_{K=0}^{n} (-1)^k b_k\, \omega^k T_k(\zeta) \qquad & B_2 &= \sum_{K=0}^{n} (-1)^k b_k\, \omega^k U_k(\zeta) \\ C_1 &= \sum_{K=0}^{n} (-1)^k c_k\, \omega^k T_k(\zeta) \qquad & C_2 &= \sum_{K=0}^{n} (-1)^k c_k\, \omega^k U_k(\zeta) \end{aligned} \tag{4-14}$$

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$$[illegible] = \beta [illegible] C_1 [illegible] + D_2 F [illegible]$$

$$[illegible] = \beta [illegible] + D_1 F_2 [illegible]$$

$$[illegible] = (C_1 D_2 - C_2 D_1) [illegible] + [illegible] C_2 [illegible] + [illegible] D_1 [illegible]$$

$$\begin{matrix} C_1 & C_2 \\ D_1 & D_2 \end{matrix} \quad \begin{vmatrix} C & D \\ D & E \end{vmatrix} \quad [illegible]$$

[illegible]

$$D_1 = \sum_{k=0}^{n} (-1)^k d_k\, \omega^k T_k(\zeta) \qquad D_2 = \sum_{k=0}^{n} (-1)^k d_k\, \omega^k U_k(\zeta)$$

$$F_1 = \sum_{k=0}^{n} (-1)^k f_k\, \omega^k T_k(\zeta) \qquad F_2 = \sum_{k=0}^{n} (-1)^k f_k\, \omega^k U_k(\zeta)$$

Eliminating α from Equations (4-12) and (4-13) we obtain:

$$\frac{F_1 + C_1\beta}{B_1 + D_1\beta} = \frac{F_2 + C_2\beta}{B_2 + D_2\beta} = \alpha \tag{4-15}$$

Which may be written:

$$\Delta_{CD}\,\beta^2 + (\Delta_{FD} + \Delta_{CB})\,\beta + \Delta_{\cancel{FD}\ FB} = 0 \tag{4-16}$$

Where:

$$\Delta_{CD} = \begin{vmatrix} C_1 & C_2 \\ D_1 & D_2 \end{vmatrix}, \quad \Delta_{FD} = \begin{vmatrix} F_1 & F_2 \\ D_1 & D_2 \end{vmatrix}, \text{ etc.}$$

Equation (4-16) has the solutions:

$$\beta = \frac{-(\Delta_{FD} + \Delta_{CB}) \pm \sqrt{(\Delta_{FD} + \Delta_{CB})^2 - 4\Delta_{CD}\Delta_{\cancel{FD}\ FB}}}{2\Delta_{CD}} \tag{4-17}$$

Let us define:

$$R_\beta^2 = (\Delta_{FD} + \Delta_{CB})^2 - 4\Delta_{CD}\Delta_{FD} \tag{4-18}$$

This permits us to write Equation (4-17) as:

$$\beta = \frac{1}{2\Delta_{CD}}\left[-(\Delta_{FD} + \Delta_{CB}) \pm R_\beta\right] \tag{4-19}$$

Proceeding in a similar fashion we may eliminate β from Equations (4-12) and (4-13) to obtain:

$$\frac{F_1 + B_1\alpha}{C_1 + D_1\alpha} = \frac{F_2 + B_2\alpha}{C_2 + D_2\alpha} = \beta \tag{4-20}$$

As before, we may rewrite this as

$$\Delta_{BD}\,\alpha^2 + (\Delta_{BC} + \Delta_{FD})\,\alpha + \Delta_{FC} = 0 \tag{4-21}$$

Solving Equation (4-21) for   we obtain:

$$\alpha = \frac{1}{2\Delta_{BD}} \left[-(\Delta_{FD} + \Delta_{BC}) \pm R_\alpha \right] \tag{4-22}$$

Where we have defined:

$$R_\alpha^2 = (\Delta_{BC} + \Delta_{FD})^2 - 4\Delta_{BD}\Delta_{FC} \tag{4-23}$$

We will now show that $R_\alpha^2 = R_\beta^2$

From Equation (4-23) we write:

$$R_\alpha^2 = (B_1C_2 - B_2C_1 + D_2F_1 - D_1F_2)^2 - 4(F_1C_2 - F_2C_1)(B_1D_2 - D_1B_2) \tag{4-24}$$

Expanding Equation (4-24) we obtain:

$$\begin{aligned} R_\alpha^2 = {} & B_1^2C_2^2 + B_2^2\,C_1^2 + D_2^2F_1^2 + D_1^2F_2^2 + 4\,B_1C_1D_2F_2 + 4\,B_2C_2D_1F_1 - 2\,B_1B_2C_1C_2 - \\ & 2\,B_1C_2D_2F_1 - 2\,B_1C_2D_1F_2 - 2\,B_2C_1D_2F_1 - 2\,B_2C_1D_1F_2 - \\ & 2\,D_1D_2F_1F_2 \end{aligned} \tag{4-25}$$

Expanding Equation (4-18) for R_β^2 we obtain the same expression as in the right hand side of Equation (4-25). Thus:

$$R_\beta^2 = R_\alpha^2 \;\hat{=}\; R^2 \tag{4-26}$$

Thus we may state from Equations (4-19) and (4-22):

$$\alpha = \frac{1}{2\Delta_{BD}} \left[-(\Delta_{FD} + \Delta_{BC}) \pm R \right] \tag{4-27}$$

$$\beta = \frac{1}{2\Delta_{CD}} \left[-(\Delta_{FD} + \Delta_{CB}) \pm R \right] \tag{4-28}$$

Noting that $\Delta_{BC} = -\Delta_{CB}$ we may rewrite Equations (4-27) and (4-28) as:

$$\alpha = \frac{1}{2\Delta_{BD}} \left[-(\Delta_{FD} + \Delta_{BC}) + (-1)^k R \right] ; \quad k = \begin{cases} 0 \\ 1 \end{cases} \tag{4-29}$$

$$\beta = \frac{1}{2\Delta_{CD}} \left[-(\Delta_{FD} + \Delta_{BC}) + (-1)^m R \right] ; \quad m = \begin{cases} 0 \\ 1 \end{cases} \tag{4-30}$$

Equations (4-29) and (4-30) define four possible solution pairs depending on the values of k and m.

Let us now determine which of these possible solutions are actually solutions to Equations (4-12) and (4-13). Multiplying Equation (4-12) by D_2 and Equation (4-13) by $-D_1$ and adding we obtain:

$$\Delta_{BD}\alpha + \Delta_{CD}\beta + \Delta_{FD} = 0 \tag{4-31}$$

Equations (4-31) must be satisfied by any of the pairs (4-30) which are to be solutions to Equations (4-12) and (4-13). Substituting Equations (4-29) and (4-30) into Equation (4-31) we obtain:

$$-\frac{\Delta_{FD}}{2} - \frac{\Delta_{BC}}{2} + \frac{(-1)^k R}{2} - \frac{\Delta_{FD}}{2} + \frac{\Delta_{BC}}{2} + \frac{(-1)^m R}{2} + \Delta_{FD} = 0 \tag{4-32}$$

Which after simplification becomes:

$$(-1)^k + (-1)^m = 0 \tag{4-33}$$

In order for (4-33) to be satisfied it is necessary that

$$k \neq m \tag{4-34}$$

Thus we may rewrite Equations (4-29) and (4-30) as:

$$\alpha_k = \frac{1}{2\Delta_{BD}} [-(\Delta_{FD} + \Delta_{BC}) + (-1)^k R] \tag{4-35}$$

$$\beta_k = \frac{1}{2\Delta_{CD}} [-(\Delta_{FD} + \Delta_{BC}) + (-1)^{k+1} R] \tag{4-36}$$

Where k = 0 and k = 1 generate the two solution pairs of Equations (4-12) and (4-13).

Equations (4-35) and (4-36) may be used to program a digital computer to generate the constant zeta and omega loci. The basic steps to be programmed are:

Constant Zeta Loci

1. Fix $\zeta = \zeta_i$. This makes Equations (4-14) functions only of the independent variable omega.

2. Increment omega and for each value of omega compute the solution pairs (α_k, β_k); k = 0, 1 using Equations (4-35) and (4-36).

3. Connect all solutions (α_0, β_0) in the order of increasing omega. This generates one branch of the constant zeta locus for $\zeta = \zeta_i$.

4. Connect all solutions (α_1, β_1) in the order of increasing omega. This generates the other branch of the constant zeta locus for $\zeta = \zeta_i$.

5. Repeat steps 1-4 for $\zeta = \zeta_{i+1}$, $\zeta = \zeta_{i+2}$, etc. until as many constant zeta loci are obtained as desired.

Constant Omega Loci

1. Fix $\omega = \omega_j$. Thus makes Equations (4-14) functions only of the independent variable zeta.

2. Increment zeta and for each value of zeta compute the solution pairs (α_k, β_k); k = 0, 1 using Equations (4-35) and (4-36).

3. Connect all solutions (α_0, β_0) in the order of increasing zeta. This generates one branch of the constant omega locus for $\omega = \omega_j$.

4. Connect all solutions (α_1, β_1) in the order of increasing zeta. This generates the other branch of the constant omega locus for $\omega = \omega_j$.

5. Repeat steps 1-4 above for $\omega = \omega_{j+1}$, $\omega = \omega_{j+2}$, etc. until as many constant omega loci are obtained as desired.

5. Non-Parametric Equations for the Constant Zeta and Omega Loci.

In this section procedures are presented which show how the equations of the constant zeta and omega loci, regardless of the form of the coefficients a_k, may be reduced to non-parametric form.

As developed in Section 4, the parameter plane equations for coefficients $a_k = b_k \alpha + c_k \beta + d_k \alpha\beta + f_k$ take the form:

$$\alpha = A(\zeta, \omega) \qquad \beta = B(\zeta, \omega) \tag{5-1}$$

The constant zeta and omega loci are obtained from these equations by fixing one of the variables, zeta or omega, and considering the other variable as independent. Varying this independent variable over the range of interest generates the desired locus. Our objective in this section is to obtain the equations of the constant zeta or omega loci from (5-1) by eliminating omega or zeta, respectively. Achieving this will produce the non-parametric equations of the desired loci:

$$Z(\alpha, \beta, \zeta) = 0 \tag{5-2}$$

$$W(\alpha, \beta, \omega) = 0 \tag{5-3}$$

Two theorems, basic to this development, are proved in Appendices I and II. These are:

<u>Theorem I</u>

The Chebyshev functions $U_k(\zeta)$ obey the following relationship:

$$U_i(\zeta)\, U_{j-1}(\zeta) - U_{i-1}(\zeta)\, U_j(\zeta) = -U_{i-j}(\zeta) \tag{5-4}$$

for all integers i and j.

Theorem II

If:

$$\sum_{k=0}^{n} (-1)^k a_k \, \omega^k U_{k-1}(\zeta) = 0$$

and if:

$$\sum_{k=0}^{n} (-1)^k a_k \, \omega^k U_k(\zeta) = 0$$

Then:

$$\sum_{k=0}^{n} (-1)^k a_k \, \omega^k U_{k-j}(\zeta) = 0 \qquad (5\text{-}5)$$

for all integers k and j.

Equations of the Constant Zeta Loci

Using Theorem II for the cases j = 0, j = n we obtain:

$$f(\omega) = \sum_{k=0}^{n} (-1)^k a_k \, \omega^k U_k(\zeta) = 0 \qquad (5\text{-}6)$$

$$g(\omega) = \sum_{k=0}^{n} (-1)^k a_k \, \omega^k U_{k-n}(\zeta) = 0 \qquad (5\text{-}7)$$

Equations (5-6) and (5-7) can be considered to be polynomial equations of order n in ω . We wish to eliminate ω from these equations in order to obtain $Z(a_k, \zeta) = 0$, which is the non-parametric representation of the constant zeta loci. This may be accomplished by employing Sylvester's Dialytic Method of Elimination (6). The basic steps are shown below for two polynomial equations of differing order in ω , from which ω is to be eliminated.

Given the polynomial equations:

$$f(\omega) = f_m \omega^m + f_{m-1} \omega^{m-1} + \ldots + f_0 = 0 \qquad (5\text{-}8)$$

$$g(\omega) = g_n \omega^n + g_{n-1} \omega^{n-1} + \ldots + g_0 = 0 \qquad (5\text{-}9)$$

The quantity ω may be eliminated from Equations (5-8) and (5-9) by the

following sequence of operations:

1. Multiply (5-8) by ω^{n-1}, ω^{n-2}, ..., 1 in turn.

2. Multiply (5-9) by ω^{m-1}, ω^{m-2}, ..., 1 in turn.

3. Form the determinant of the coefficient matrix of the resulting set of (n+m) equations which are linear and homogeneous in the (n+m) quantities ω^{m+n-1}, ω^{m+n-2}, ..., ω, 1.

4. This determinant, when forced equal to zero, establishes the relation among the f_k and g_k necessary if Equations (5-8) and (5-9) are to have common roots.

The resulting determinant has the form:

$$Z(f_k, g_k) = \begin{vmatrix} f_m & \cdot & \cdot & \cdot & f_2 & f_1 & f_0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & f_m & \cdot & \cdot & \cdot & f_2 & f_1 & f_0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & f_m & \cdot & \cdot & \cdot & f_2 & f_1 & f_0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & f_m & \cdot & \cdot & f_2 & f_1 & f_0 \\ g_n & \cdot & \cdot & \cdot & g_2 & g_1 & g_0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & g_n & \cdot & \cdot & \cdot & g_2 & g_1 & g_0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & g_n & \cdot & \cdot & \cdot & g_2 & g_1 & g_0 \end{vmatrix} = 0 \qquad (5\text{-}10)$$

In the case of Equations (5-6) and (5-7), the constant zeta loci can be obtained from Equation (5-10) by making the following substitutions:

$$f_k = (-1)^k a_k U_k(\zeta) \qquad (5\text{-}11)$$

$$g_k = (-1)^k a_k U_{k-n}(\zeta) \qquad (5\text{-}12)$$

The following example will demonstrate the technique:

Example 2

Consider again the symmetrical parallel-tee network shown in Figure 2, Section 3. For the zeros of the transfer of this network, Equations (5-6) and (5-7) become:

$$-\alpha\beta\,\omega^3 U_3(\zeta) + 2\beta\,\omega^2 U_2(\zeta) - 2\beta U_1(\zeta) + U_0(\zeta) = 0 \tag{5-13}$$

$$-\alpha\beta\,\omega^3 U_0(\zeta) + 2\beta\,\omega^2 U_{-1}(\zeta) - 2\beta U_{-2}(\zeta) + U_{-3}(\zeta) = 0 \tag{5-14}$$

Multiplying these equations by ω^2, ω, and 1 and forming the determinant $Z(\alpha, \beta, \zeta)$, we obtain:

$$Z(\alpha,\beta,\zeta) = \begin{vmatrix} -\alpha\beta U_3 & +2\beta U_2 & -2\beta U_1 & +U_0 & 0 & 0 \\ 0 & -\alpha\beta U_3 & +2\beta U_2 & -2\beta U_1 & +U_0 & 0 \\ 0 & 0 & -\alpha\beta U_3 & +2\beta U_2 & -2\beta U_1 & +U_0 \\ -\alpha\beta U_0 & +2\beta U_{-1} & -2\beta U_{-2} & +U_{-3} & 0 & 0 \\ 0 & -\alpha\beta U_0 & +2\beta U_{-1} & -2\beta U_{-2} & +U_{-3} & 0 \\ 0 & 0 & -\alpha\beta U_0 & +2\beta U_{-1} & -2\beta U_{-2} & +U_{-3} \end{vmatrix} = 0 \tag{5-15}$$

Substituting the appropriate functions of zeta for the U_k in Equation (5-15) and expanding, we obtain the equation for the constant zeta loci:

$$-\alpha\beta^3(4\zeta^2-1)^3[(-32\alpha\zeta^2+16)\beta^2 + (64\alpha\zeta^4+16\alpha\zeta^2 - 32\zeta^2 - 8\alpha)\beta + (-64\alpha^2\zeta^6 + 48\alpha^2\zeta^4 - 12\alpha^2\zeta^2 + \alpha^2)] = 0 \tag{5-16}$$

When the two equations (5-8) and (5-9) are of the same order so that m = n, Sylvester's (2n x 2n) determinant can be reduced to an (n x n) determinant by using Bezout's Method of Elimination (7). To demonstrate Bezout's method, consider Equations (5-8) and (5-9) with m = n = 3. This produces the equations:

$$f(\omega) = f_3\,\omega^3 + f_2\,\omega^2 + f_1\,\omega^1 + f_0 = 0 \tag{5-17}$$

$$g(\omega) = g_3\,\omega^3 + g_2\,\omega^2 + g_1\,\omega^1 + g_0 = 0 \tag{5-18}$$

Then:

$$f_3 g - g_3 f = 0 \tag{5-19}$$

$$(f_3\omega + f_2)g - (g_3\,\omega + g_2)f = 0 \tag{5-20}$$

$$(f_3\,\omega^2 + f_3\,\omega + f_1)g - (g_3\,\omega^2 + g_2\,\omega + g_1)f = 0 \tag{5-21}$$

can be written in matrix form:

$$\begin{bmatrix} (f_3g_2) & (f_3g_1) & (f_3g_0) \\ (f_3g_1) & ((f_3g_0) + (f_2g_1)) & (f_2g_0) \\ (f_3g_0) & (f_2g_0) & (f_1g_0) \end{bmatrix} \begin{bmatrix} \omega^2 \\ \omega \\ 1 \end{bmatrix} = \underline{0} \tag{5-22}$$

Where $(f_i g_j)$ denote (2 x 2) determinants:

$$(f_i g_j) = \begin{vmatrix} f_i & f_j \\ g_i & g_j \end{vmatrix} \tag{5-23}$$

In order to eliminate ω from Equations (5-17) and (5-18) by Bezout's method it is necessary to force the determinant of the coefficient matrix in Equation (5-23) to zero. Thus the constant zeta loci for the case m = n = 3 are given in non-parametric form by:

$$Z_3(f_k, g_k) = \begin{vmatrix} (f_3 g_2) & (f_3 g_1) & (f_3 g_0) \\ (f_3 g_1) & ((f_3 g_0) + (f_2 g_1)) & (f_2 g_0) \\ (f_3 g_0) & (f_2 g_0) & (f_1 g_0) \end{vmatrix} = 0 \qquad (5\text{-}24)$$

For cases where $m = n > 3$, a determinant similar to (5-24) can be developed by forming n equations as in (5-19) - (5-21). When $m = n = 4$ such a procedure gives the following determinantal equation for the constant zeta loci:

$$Z_4(\alpha, \beta, \zeta) = \begin{vmatrix} (f_4 g_3) & (f_4 g_2) & (f_4 g_1) & (f_4 g_0) \\ (f_4 g_2) & ((f_4 g_1) + (f_3 g_2)) & ((f_4 g_0) + (f_3 g_1)) & (f_3 g_0) \\ (f_4 g_1) & ((f_4 g_0) + (f_3 g_1)) & ((f_3 g_0) + (f_2 g_1)) & (f_2 g_0) \\ (f_4 g_0) & (f_3 g_0) & (f_2 g_0) & (f_1 g_0) \end{vmatrix} = 0 \qquad (5\text{-}25)$$

The following example will demonstrate how Bezout's method is applied to obtain the constant zeta loci:

<u>Example III</u>

Let us again consider the symmetrical parallel-tee network. Equations (5-13) and (5-14) yield:

$f_3 = -\alpha/\beta\, U_3 \qquad f_2 = +2\beta U_2 \qquad f_1 = -2\beta U_1 \qquad f_0 = +U_0$

$g_3 = -\alpha/\beta\, U_0 \qquad g_2 = +2\beta U_{-1} \qquad g_1 = -2\beta U_{-2} \qquad g_0 = +U_{-3}$

Substituting the appropriate functions of zeta into these expressions and forming the (2 x 2) determinants for the $(f_i g_j)$ we obtain:

$$(f_3 g_2) = +2\alpha\beta^2(4\zeta^2-1)$$

$$(f_3 g_0) = -4\alpha\beta^2\zeta(4\zeta^2-1)$$

$$(f_3 g_0) = (4\alpha\beta\zeta^2 - \alpha\beta)(4\zeta^2-1)$$

$$(f_2 g_1) = 4\beta^2(4\zeta^2-1)$$

$$(f_2 g_0) = -4\beta\zeta(4\zeta^2-1)$$

$$(f_1 g_0) = +2\beta(4\zeta^2-1)$$

When these are substituted into Equation (5-24) we obtain:

$$Z_3(\alpha, \beta, \zeta) = \alpha\beta^3(4\zeta^2-1)^3 \begin{vmatrix} +2\beta & -4\beta\zeta & (4\zeta^2-1) \\ -4\alpha\beta\zeta & (4\alpha\zeta^2 + 4\beta - \alpha) & -4\zeta \\ (4\alpha\zeta^2 - \alpha) & -4\zeta & +2 \end{vmatrix} = 0$$

Which when expanded gives the constant zeta loci in non-parametric form:

$$Z_3(\alpha, \beta, \zeta) = \alpha\beta^3(4\zeta^2-1)^3 \; [(-32\alpha\zeta^2+16)\beta^2 + (64\alpha\zeta^4+16\alpha\zeta^2-32\zeta^2 - 8\alpha)\beta + (-64\alpha^2\zeta^2 + 48\alpha^2\zeta^4 - 12\alpha^2\zeta^2 + \alpha^2)] = 0$$

Observe that the determinant obtained by Bezout's method has the opposite sign from that obtained by Sylvester's method. Sylvester's method produces the resultant of the functions $f(\omega)$ and $g(\omega)$, while Bezout's method produces the negative of the resultant (8). Since multiplying both sides of the determinantal equations by -1 does not introduce extraneous common factors of $f(\omega)$ and $g(\omega)$, this sign difference can be ignored.

It would appear at first that Bezout's method would be easier to apply

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than Sylvester's method because of the reduction in the order of the determinants that it affords. However, Sylvester's determinant has many null elements as well as functionally simpler elements than Bezout's determinant. It is the author's opinion that Sylvester's method is somewhat easier to apply.

Equations of the Constant Omega Loci

Using Theorem (II for the cases j = 0, j = 1, ..., j = (n-1) we obtain:

$$
\begin{aligned}
&\sum_{k=0}^{n} (-1)^k a_k \omega^k U_k(\zeta) = 0 \\
&\sum_{k=0}^{n} (-1)^k a_k \omega^k U_{k-1}(\zeta) = 0 \\
&\quad\vdots \\
&\sum_{k=0}^{n} (-1)^k a_k \omega^k U_{k-(n-1)}(\zeta) = 0
\end{aligned}
\qquad (5\text{-}26)
$$

These equations may be written:

$$
\begin{aligned}
&(-1)^0 a_0 \omega^0 U_0 + (-1)^1 a_1 \omega^1 U_1 + \dots + (-1)^n a_n \omega^n U_n = 0 \\
&(-1)^0 a_0 \omega^0 U_{-1} + (-1)^1 a_1 \omega^1 U_0 + \dots + (-1)^n a_n \omega^n U_{n-1} = 0 \\
&\quad\vdots \\
&(-1)^0 a_0 \omega^0 U_{-(n-1)} + (-1)^1 a_1 \omega^1 U_{-(n-2)} + \dots + (-1)^n a_n \omega^n U_1 = 0
\end{aligned}
\qquad (5\text{-}27)
$$

Making use of the relation $U_{-n}(\zeta) = -U_n(\zeta)$ and $U_0 = 0$ and rearranging the set of equations (5-27) we obtain:

$$
\begin{bmatrix}
(-1)^1 a_1 \omega^1 & \dots & (-1)^n a_n \omega^n \\
[(-1)^2 a_2 \omega^2 - (-1)^0 a_0 \omega^0] & \dots & 0 \\
[(-1)^n a_n \omega^n - (-1)^{n-2} a_{n-2} \omega^{n-2}] & \dots & 0 \\
(-1)^{n-1} a_{n-1} \omega^{n-1} & \dots & (-1)^0 a_0 \omega^0
\end{bmatrix}
\begin{bmatrix}
U_1 \\ U_2 \\ U_3 \\ U_{n-1}
\end{bmatrix}
= \underline{0} \qquad (5\text{-}28)
$$

The set of equations (5-28) are linear and homogeneous in the quantities U_k. Thus, for a non-trivial solution to exist, the determinant of the coefficient matrix in (5-28) must vanish. This condition specifies the relation necessary among the a_k and ω which specifies the constant omega loci:

$$W(a_k,\omega) = \begin{vmatrix} (-1)^1 a_1\omega^1 & \cdots & (-1)^n a_n\omega^n \\ [(-1)^2 a_2\omega^2-(-1)^0\omega^0] & \cdots & 0 \\ [(-1)^n a_n\omega^n-(-1)^{n-2}\omega^{n-2}] & \cdots & 0 \\ (-1)^{n-1} a_{n-1}\omega^{n-1} & \cdots & (-1)^0 a_0\omega^0 \end{vmatrix} = 0 \tag{5-29}$$

Example IV

Let us apply this procedure to obtain the constant omega locus for the zeros of the symmetrical parallel-tee network. From Equation (3-2) the zeros of the transfer function for this network are the roots of:

$$\alpha\beta p^3 + 2\beta p^2 + 2\beta p + 1 = 0 \tag{5-30}$$

Applying theorem II, we may write as in Equation (5-26):

$$\begin{aligned} (-1)^1(2\beta)\omega^1 U_1 + (-1)^2(2\beta)\omega^2 U_2 + (-1)^3(\alpha\beta)\omega^3 U_3 &= 0 \\ (-1)^0(1)\omega^0 U_{-1} + (-1)^2(2\beta)\omega^2 U_1 + (-1)^3(\alpha\beta)\omega^3 U_2 &= 0 \\ (-1)^0(1)\omega^0 U_{-2} + (-1)^1(2\beta)\omega^1 U_{-1} + (-1)^3(\alpha\beta)\omega^3 U_1 &= 0 \end{aligned} \tag{5-31}$$

Using the relation $U_{-k} = - U_k$ these equations can be written:

$$\begin{bmatrix} (-1)^1(2\beta)\omega^1 & (-1)^2(2\beta)\omega^2 & (-1)^3(\alpha\beta)\omega^3 \\ [(-1)^2(2\beta)\omega^2-(-1)^0(1)\omega^0] & (-1)^2(\alpha\beta)\omega^3 & 0 \\ [(-1)^3(\alpha\beta)\omega^3-(-1)^1(2\beta)\omega^1] & -(-1)^0(1)\omega^0 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \underline{0}$$

Setting the determinant of the coefficient matrix to zero we obtain:

$$W(\alpha,\beta,\omega) = \begin{vmatrix} -2\beta\omega & +2\beta\omega^2 & -\alpha\beta\omega^3 \\ (2\beta\omega^2-1) & -\alpha\beta\omega^3 & 0 \\ (2\beta\omega-\alpha\beta\omega^3) & -1 & 0 \end{vmatrix} = 0 \qquad (5\text{-}32)$$

Which, when expanded becomes:

$$W(\alpha,\beta,\omega) = \alpha\beta\omega^3(\alpha^2\beta^2\omega^6 - 2\alpha\beta^2\omega^4 + 2\beta\omega^2 - 1) = 0 \qquad (5\text{-}33)$$

Thus the equation

$$(\alpha^2\omega^6 - 2\alpha\omega^4)\beta^2 + (+2\omega^2)\beta - 1 = 0 \qquad (5\text{-}34)$$

gives the constant omega loci except where $\alpha = 0$, $\beta = 0$ or $\omega = 0$.

6. Equations of the Constant Bandwidth Curves

In this section, equations will be developed for displaying curves of constant bandwidth upon the parameter plane.

Let us define what is meant by a constant bandwidth curve: "A constant bandwidth curve for $G(j\omega_b) = M$ is a curve drawn upon the parameter plane which specifies the relation between the parameters necessary if the transfer function G(s), which is a function of the parameters, is to have magnitude M at the real frequency ω_b."

Once such curves are obtained for selected values of M, the frequency response $G(j\omega)$ may be sketched. Alternately the constant bandwidth curve corresponding to some M and ω_b specification can be drawn, and from this curve values of the parameters can be found which will guarantee that the specification is met.

Consider the following rational transfer function G(s):

$$G(s) = \frac{q_m s^m + q_{m-1} s^{m-1} + \ldots + q_1 s + q_o}{r_n s^n + r_{n-1} s^{n-1} + \ldots + r_1 s + r_0} = \underline{Q(s)} \tag{6-1}$$

Where the coefficients of Q(s) and R(s) are of the form:

$$q_p = e_p \alpha + f_p \beta + g_p \alpha\beta + h_p \; ; \; p = 0, 1, \ldots m \tag{6-2}$$

$$r_k = a_k \alpha + b_k \beta + c_k \alpha\beta + d_k \; ; \; k = 0, 1, \ldots n \tag{6-3}$$

Thus, G(s) may be written:

$$G(s) = \frac{\sum_{p=0}^{m} (e_p \alpha + f_p \beta + g_p \alpha\beta + h_p) \; s^p}{\sum_{k=0}^{n} (a_k \alpha + b_k \beta + c_k \alpha\beta + d_k) \; s^k} \tag{6-4}$$

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Consider now only real frequency, $s = j\omega$, this can be written:

$G(j\omega) =$

$$\frac{\displaystyle\sum_{\substack{p=0\\ \text{even}}}^{m} (-1)^{\frac{1}{2}p}\omega^{p}(e_p\alpha + f_p\beta + g_p\alpha\beta + h_p) + j\sum_{\substack{p=1\\ \text{odd}}}^{m} (-1)^{\frac{1}{2}(p-1)}\omega^{p}(e_p\alpha + f_p\beta + g_p\alpha\beta + h_p)}{\displaystyle\sum_{\substack{k=0\\ \text{even}}}^{n} (-1)^{\frac{1}{2}k}\omega^{k}(a_k\alpha + b_k\beta + c_k\alpha\beta + d_k) + j\sum_{\substack{k=1\\ \text{odd}}}^{n} (-1)^{\frac{1}{2}(k-1)}\omega^{k}(a_k\alpha + b_k\beta + c_k\alpha\beta + d_k)} \tag{6-5}$$

Now let $\omega = \omega_b$, some fixed value of ω, and also define:

$$A_r = \sum_{\substack{k=0\\ \text{even}}}^{n} (-1)^{\frac{1}{2}k}\,\omega_b^k a_k \text{ ; etc. for } B_r,\ C_r,\ D_r \tag{6-6}$$

$$A_i = \sum_{\substack{k=1\\ \text{odd}}}^{n} (-1)^{\frac{1}{2}(k-1)}\,\omega_b^k a_k \text{ ; etc. for } B_i,\ C_i,\ D_i \tag{6-7}$$

$$E_r = \sum_{\substack{p=0\\ \text{even}}}^{m} (-1)^{\frac{1}{2}p}\,\omega_b^p e_p \text{ ; etc. for } F_r, G_r,\ H_r \tag{6-8}$$

$$E_i = \sum_{\substack{p=1\\ \text{odd}}}^{m} (-1)^{\frac{1}{2}(p-1)}\,\omega_b^p e_p \text{ ; etc. for } F_i,\ G_i,\ H_i \tag{6-9}$$

Using definitions (6-6) thru (6-9), Equation (6-5) can be written:

$$G(j\omega_b) = \frac{(\alpha E_r + \beta F_r + \alpha\beta G_r + H_r) + j(\alpha E_i + \beta F_i + \alpha\beta G_i + H_i)}{(\alpha A_r + \beta B_r + \alpha\beta C_r + D_r) + j(\alpha A_i + \beta B_i + \alpha\beta C_i + D_i)} \tag{6-10}$$

Using (6-10) and defining the magnitude of $G(j\omega_b)$ as M we obtain:

$$M^2 = |G(j\omega_b)|^2 = \frac{(\alpha E_r + \beta F_r + \alpha\beta G_r + H_r)^2 + (\alpha E_i + \beta F_i + \alpha\beta G_i + H_i)^2}{(\alpha A_r + \beta B_r + \alpha\beta C_r + D_r)^2 + (\alpha A_i + \beta B_i + \alpha\beta C_i + D_i)^2} \qquad (6\text{-}11)$$

Which when expanded becomes:

$$M^2 = \frac{\phi(\alpha,\beta)}{\theta(\alpha,\beta)} \qquad (6\text{-}12)$$

Where:

$$\begin{aligned}\phi(\alpha,\beta) = {}& \alpha^2\beta^2(G_r^2 + G_i^2) + 2\alpha^2\beta(E_rG_r + E_iG_i) + 2\alpha\beta^2(F_rG_r + F_iG_i) + \\ & \alpha^2(E_r^2 + E_i^2) + \beta^2(F_r^2 + F_i^2) + 2\alpha\beta(E_rF_r + E_iF_i) + \\ & 2\alpha(E_rH_r + E_iH_i) + 2\beta(F_rH_r + F_iH_i) + (H_r^2 + H_i^2)\end{aligned} \qquad (6\text{-}13)$$

$$\begin{aligned}\theta(\alpha,\beta) = {}& \alpha^2\beta^2(C_r^2 + C_i^2) + 2\alpha^2\beta(A_rC_r + A_iC_i) + 2\alpha\beta^2(B_rC_r + B_iC_i) + \\ & \alpha^2(A_r^2 + A_i^2) + \beta^2(B_r^2 + B_i^2) + 2\alpha\beta(A_rB_r + A_iB_i) + \\ & 2\alpha(A_rD_r + A_iD_i) + 2\beta(B_rD_r + B_iD_i) + (D_r^2 + D_i^2)\end{aligned} \qquad (6\text{-}14)$$

Equation (6-12) can be written:

$$\phi(\alpha,\beta) - (M^2\theta(\alpha,\beta)) = 0 \qquad (6\text{-}15)$$

Substituting (6-13) and (6-14) into (6-15) and expressing the resulting equation in the form of a quadratic in β we obtain:

$$(M^2R_1 - R_2)\beta^2 + (M^2W_1 - W_2)\beta + (M^2V_1 - V_2) = 0 \qquad (6\text{-}16)$$

Where:

$$R_1 = (B_r^2 + B_i^2) + 2\alpha(B_rC_r + B_iC_i) + \alpha^2(C_r^2 + C_i^2)$$

$$R_2 = (F_r^2 + F_i^2) + 2\alpha(F_rG_r + F_iG_i) + \alpha^2(G_r^2 + G_i^2)$$

$$W_1 = 2(B_rD_r + B_iD_i) + 2\alpha(A_rB_r + A_iB_i + C_rD_r + C_iD_i) + 2\alpha^2(A_rC_r + A_iC_i)$$

$$W_2 = 2(F_rH_r + F_iH_i) + 2\alpha(E_rF_r + E_iF_i + G_rH_r + G_iH_i) + 2\alpha^2(E_rG_r + E_iG_i)$$

$$V_1 = (D_r^2 + D_i^2) + 2\alpha(A_rD_r + A_iD_i) + \alpha^2(A_r^2 + A_i^2)$$

$$V_2 = (H_r^2 + H_i^2) + 2\alpha(E_rH_r + E_iH_i) + \alpha^2(E_r^2 + E_i^2)$$

We may now solve the quadratic equation (6-16) for β as a function of α, ω_b, and M. This solution is:

$$\beta_{1,2} = \frac{-(M^2W_1 - W_2) \pm \sqrt{(M^2W_1 - W_2)^2 - 4(M^2R_1 - R_2)(M^2V_1 - V_2)}}{2(M^2R_1 - R_2)} \qquad (6\text{-}17)$$

The constant bandwidth curve corresponding to some desired $M = M_0$ and $\omega_b = \omega_{b0}$ can be computed from (6-17) by a digital computer. The procedure is to fix M and ω_b in (6-17) at the desired values, increment α over the range of interest, and at each value of α to compute the corresponding values of β. The two roots of (6-17) give the two branches of the desired constant bandwidth curve. The reader is referred to section 10 where constant bandwidth curves are computed and discussed for a loaded and null adjusted parallel-tee network.

7. Equations of the Constant Q Loci for Second Order Systems

Recent interest in the Q of second order RC networks, accompanied by difficulties in applying the conventional definitions of Q, led Morris (4) to define Q mathematically as a parameter of the characteristic equation. He rearranged the second order characteristic equation into the form (7-1) which defines Q:

$$T^2 s^2 + \frac{T}{Q} s + 1 = 0 \tag{7-1}$$

Dutta-Roy (9) has pointed out that Q so defined is equivalent to an earlier definition proposed by Bolle (10). Bolle's definition for the second order characteristic equation:

$$a_2 s^2 + a_1 s + a_0 = 0 \tag{7-2}$$

is:

$$Q = \frac{(a_0 a_2)^{\frac{1}{2}}}{a_1} \tag{7-3}$$

In this section equations will be obtained which will permit loci of constant Q to be displayed upon the parameter plane for variable parameter second order characteristic equations of the form (7-2). It is assumed that the variable parameters appear in the coefficients of (7-2) in the following manner:

$$a_k = b_k \alpha + c_k \beta + d_k \alpha/\beta + f_k \tag{7-4}$$

$$k = 0,1, \text{ and } 2$$

Squaring both sides of Equation (7-3) and rearranging we obtain:

$$Q^2 a_1^{2} - a_0 a_2 = 0 \tag{7-5}$$

Substituting (7-4) into (7-5) and writing the resulting equation in the

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form of a quadratic in , we obtain:

$$[(Q^2d_1^2-d_0d_2)\alpha^2 + (2Q^2c_1d_1-c_0d_2-c_2d_0)\alpha + (Q^2c_1^2-c_0c_2)]\beta^2 +$$

$$[(2Q^2b_1d_1-b_0d_2-b_2d_0)\alpha^2 + (2Q^2b_1c_1+2Q^2d_1f_1-b_2c_0-b_0c_2-d_2f_0-d_0f_2)\alpha +$$

$$(2Q^2c_1f_1-c_2f_0-c_0f_2)]\beta +$$

$$[(Q^2b_1^2-b_0b_2)\alpha^2 + (2Q^2b_1f_1-b_2f_0-b_0f_2)\alpha + (Q^2f_1^2-f_0f_2)] = 0 \qquad (7\text{-}6)$$

The desired constant Q loci can be obtained from (7-5) by use of a digital computer. The procedure is as follows:

1. Fix Q in (7-5) at a value for which the locus is desired.
2. Increment α over the range of interest.
3. For each value of α, solve (7-5) by the quadratic formula.
4. The two solutions of (7-5), β_1 and β_2, plotted as ordinates on the parameter plane with α as abscissa define the two branches of the desired constant Q locus.
5. Repeat steps 1-4 for each value of Q for which a locus of constant Q is desired.

For examples of this procedure the reader is referred to Section 10 where several loci of constant Q are obtained for a variable parameter, loaded, and null-adjusted parallel-tee network.

8. Sensitivity Analysis by Parameter Plane Techniques.

a. Introduction

The polynomial equation

$$\sum_{k=0}^{n} a_k s^k = 0 \tag{8-1}$$

where : $a_k = f(q_1, q_2, \ldots, q_m)$

has n roots. Each of these roots is a function of the parameters q_1, q_2, $\ldots, q_r, \ldots q_m$. It is the objective of this section to show how each of these roots vary as the parameters q_r change.

Let us denote the i^{th} complex root pair of (8-1) by:

$$r_i = -\zeta_i \omega_i \pm j\, \omega_i \sqrt{1-\zeta^2} \tag{8-2}$$

and the j^{th} real root of (8-1) by:

$$r_j = \sigma_j \tag{8-3}$$

Let us now define two types of sensitivity; macroscopic sensitivity and microscopic sensitivity.

Macroscopic sensitivity is the sensitivity of the roots (8-2) and (8-3) to large changes in the parameters q_r. These sensitivities may be defined by:

$$\begin{aligned}
S_{i,r}^{\zeta_i} &= (\Delta \zeta_i)/(\Delta q_r) \qquad \Delta q_i = 0,\ i \neq r \\
S_{i,r}^{\omega_i} &= (\Delta \omega_i)/(\Delta q_r) \qquad \Delta q_i = 0,\ i \neq r \\
S_{j,r}^{\sigma_j} &= (\Delta \sigma_j)/(\Delta q_r) \qquad \Delta q_i = 0,\ i \neq r
\end{aligned} \tag{8-4}$$

$S_{i,r}^{\zeta}$ denotes the change in the damping ratio ζ_i of the complex root pair r_i due to a finite change in the parameter q_r. $S_{i,r}^{\omega_i}$ denotes the change in

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the undamped natural frequency of the complex root pair r_i due to a finite change in the parameter q_r. $S_{j,r}^{\sigma_j}$ denotes the change in the real root r_j due to a finite change in the parameter q_r.

Microscopic sensitivity is the sensitivity of the roots (8-2) and (8-3) to infinitesmal changes in the parameters q_r. These may be defined as:

$$s_{i,r}^{\zeta_i} = \partial \zeta_i / \partial q_r$$

$$s_{i,r}^{\omega_i} = \partial \omega_i / \partial q_r \qquad (8\text{-}5)$$

$$s_{j,r}^{\sigma_j} = \partial \sigma_j / \partial q_r$$

The interpretation of these microscopic sensitivities except that the parameter changes are infinitesmal rather than finite.

b. Obtaining Macroscopic Sensitivity from the Parameter Plane

The procedure for obtaining macroscopic sensitivity directly from the parameter plane is best demonstrated by an example.

<u>Example V</u>

Let us consider some third order polynomial equation:

$$a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0 \ ; \quad a_k = f_k(\alpha, \beta)$$

Let us also imagine that we have obtained the parameter plane for this polynomial and that it appears as shown in Figure 3: Let us presume that we have selected $\alpha = \alpha_0 = 3.8$ and $\beta = \beta_0 = 2.4$ in order to place a pair of complex roots at $\zeta = 0.2$ and $\omega = 0.7$ and that this choice results in a real root at $\sigma = -3.0$. We wish to determine the macroscopic sensitivities S_α^ζ, S_β^ζ, S_α^ω, S_β^ω, S_α^σ, and S_β^σ, for an increase of one unit in both α and β. From Figure 3 we determine:

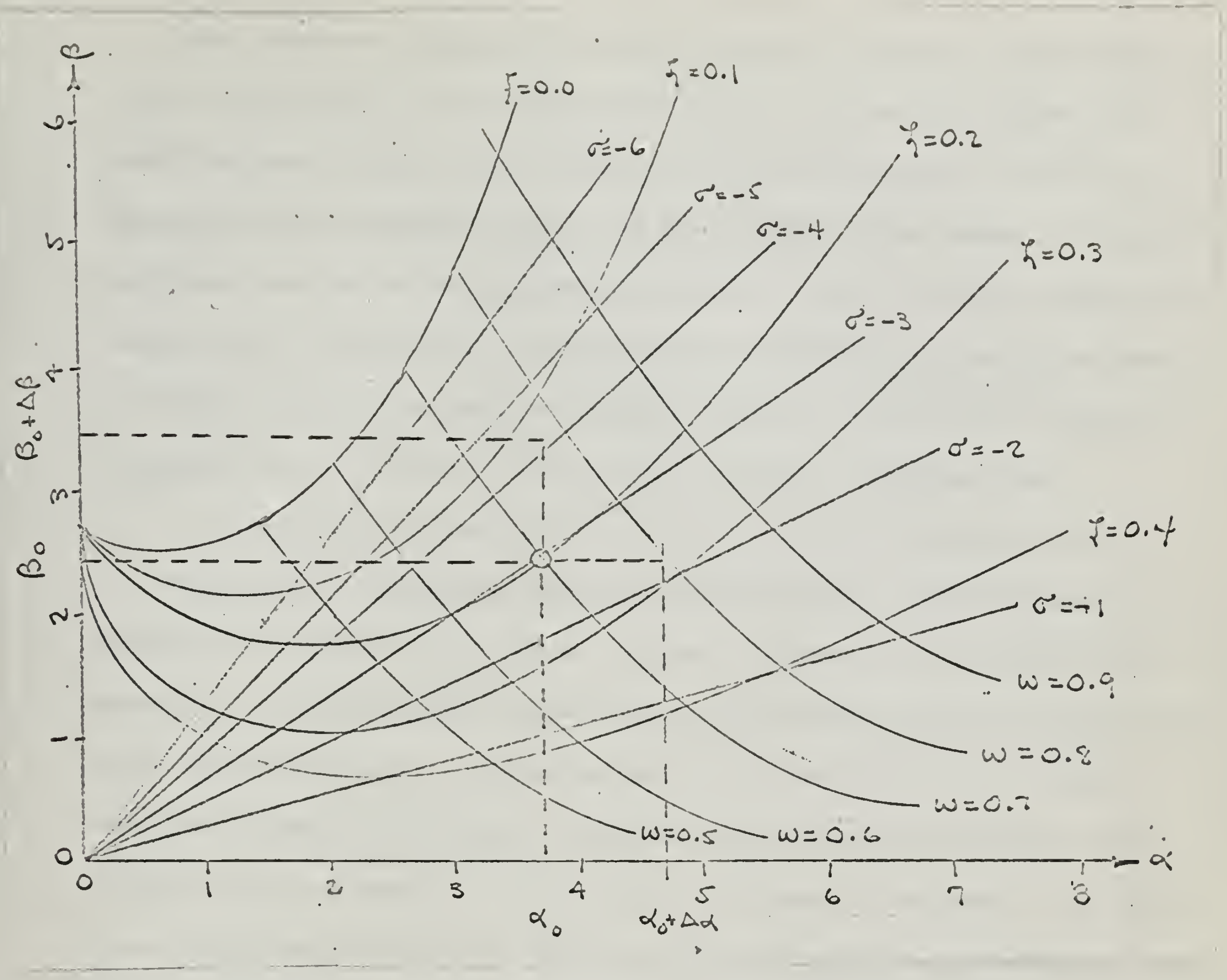

Fig. 3 Macroscopic Sensitivity from the Parameter Plane.

$$S_{\alpha}^{\zeta} = \left.\frac{\Delta\zeta}{\Delta\alpha}\right|_{\Delta\beta=0} \approx \frac{0.28-0.20}{4.80-3.80} = 0.08$$

$$S_{\beta}^{\zeta} = \left.\frac{\Delta\zeta}{\Delta\beta}\right|_{\Delta\alpha=0} = \frac{0.12-0.20}{3.40-2.40} = -0.08$$

$$S_{\alpha}^{\omega} = \left.\frac{\Delta\omega}{\Delta\alpha}\right|_{\Delta\beta=0} = \frac{0.80-0.70}{4.80-3.80} = 0.10$$

$$S_{\beta}^{\omega} = \left.\frac{\Delta\omega}{\Delta\beta}\right|_{\Delta\alpha=0} = \frac{0.77-0.70}{3.40-2.40} = -0.07$$

$$S_{\alpha}^{\sigma} = \left.\frac{\Delta\sigma}{\Delta\alpha}\right|_{\Delta\beta=0} = \frac{2.20-3.00}{4.80-3.80} = -0.80$$

$$S_{\beta}^{\sigma} = \left.\frac{\Delta\sigma}{\Delta\beta}\right|_{\Delta\alpha=0} = \frac{4.20-3.00}{3.40-2.40} = 1.2$$

More important perhaps, than these numerical values of the macroscopic sensitivities is the qualitative, but enlightening information about the sensitivity of the roots which can be obtained by careful inspection of the parameter plane. If the constant zeta, omega, and sigma loci have been drawn for constant increments in each of these quantities respectively, then one can immediately see that the most sensitive areas are those in which the loci are tightly spaced. This is like looking at a contour map of terrain- steep areas are highly sensitive areas.

c. Obtaining Microscopic Sensitivity from the Parameter Plane

Microscopic sensitivity can be obtained either by the method of Kokotovic and Siljak (5) or by an alternate method to be developed here. Kokotovic and Siljak derive relations for sensitivity from the basic parameter plane equations. The new method, to be described here, obtains sensitivity relations directly from the non-parametric equations of the constant zeta and omega loci which were developed in section 5. In those cases where the equations of these loci have already been obtained by previous calculation, sensitivity relations can be more rapidly developed from these loci equations than by the method of Kokotovic and Siljak. As background, the method of Kokotovic and Siljak will be discussed first, and then the method of obtaining sensitivity relations from the non-parametric equations of the constant zeta and omega loci will be introduced.

(1) Method of Kokotovic and Siljak

The basic parameter plane equations are:

$$f = \sum_{k=0}^{n} (-1)^{k-1} a_k \, \omega^k U_{k-1}(\zeta) = 0 \qquad (8\text{-}6)$$

$$g = \sum_{k=0}^{n} (-1)^k a_k \,\omega^k U_k(\zeta) = 0 \tag{8-7}$$

Functionally these equations can be written:

$$f(\zeta, \omega, a_k) = 0 \tag{8-8}$$

$$g(\zeta, \omega, a_k) = 0 \tag{8-9}$$

Consider that ζ and ω in (8-8) and (8-9) are implicit functions of the a_k. The a_k are in turn explicit functions of the q_r.

Thus:

$$\partial f/\partial q_i = (\partial f/\partial \omega_i)(\partial \omega_i/\partial q_i)+(\partial f/\partial \zeta_i)(\partial \zeta_i/\partial q_i)+(\partial f/\partial a_k)(\partial a_k/\partial q_i) \tag{8-10}$$

$$\partial g/\partial q_i = (\partial g/\partial \omega_i)(\partial \omega_i/\partial q_i)+(\partial g/\partial \zeta_i)(\partial \zeta_i/\partial q_i)+(\partial g/\partial a_k)(\partial a_k/\partial q_i) \tag{8-11}$$

But from (8-6) and (8-7):

$$\partial f/\partial \omega_i = (1/\omega_i) \sum_{k=0}^{n} (-1)^{k-1} k a_k \,\omega_i^k U_{k-1}(\zeta_i) \triangleq A_1^i \tag{8-12}$$

$$\partial f/\partial \zeta_i = \sum_{k=0}^{n} (-1)^{k-1} a_k \,\omega_i^k U'_{k-1}(\zeta_i) \triangleq A_2^i \tag{8-13}$$

$$\partial g/\partial \omega_i = (1/\omega_i) \sum_{k=0}^{n} (-1)^k k a_k \,\omega_i^k U_k(\zeta_i) \triangleq B_1^i \tag{8-14}$$

$$\partial g/\partial \zeta_i = \sum_{k=0}^{n} (-1)^k a_k \,\omega_i^k U'_k(\zeta_i) \triangleq B_2^i \tag{8-15}$$

Substituting (8-12) thru (8-15) into (8-10) and (8-11) we obtain:

$$\partial f/\partial q_i = (1/\omega_i)(\partial \omega_i/\partial q_r) \sum_{k=0}^{n} (-1)^{k-1} k a_k \omega_i^k U_{k-1} +$$

$$(\partial \zeta_i/\partial q_r) \sum_{k=0}^{n} (-1)^{k-1} a_k \omega_i^k U'_{k-1} +$$

$$\sum_{k=0}^{n} (\partial a_k/\partial q_r)(-1)^{k-1} \omega_i^k U_{k-1} = 0 \tag{8-16}$$

$$\partial g/\partial q_i = (1/\omega_i)(\partial \omega_i/\partial q_r) \sum_{k=0}^{n} (-1)^{k-1} k a_k \omega_i^k U_k +$$

$$(\partial \zeta_i/\partial q_r) \sum_{k=0}^{n} (-1)^{k} a_k \omega_i^k U'_k +$$

$$\sum_{k=0}^{n} (\partial a_k/\partial q_r)(-1)^{k} \omega_i^k U_k = 0 \tag{8-17}$$

Let:

$$C^i_{1,r} = \sum_{k=0}^{n} (\partial a_k/\partial q_r)\ (-1)^{k-1} \omega_i^k U_{k-1} \tag{8-19}$$

$$C^i_{2,r} = \sum_{k=0}^{n} (\partial a_k/\partial q_r)\ (-1)^{k} \omega_i^k U_k \tag{8-20}$$

We may now rewrite (8-16) and (8-17) as:

$$(A^i_1/\omega_i)\ \partial \omega_i/\partial q_r + (B^i_1)\ \partial \zeta_i/\partial q_r = -C^i_{1,r} \tag{8-21}$$

$$(A^i_2/\omega_i)\ \partial \omega_j/\partial q_r + (B^i_2)\ \partial \zeta_i/\partial q_r = -C^i_{2,r} \tag{8-22}$$

Equations (8-21) and (8-22) are simultaneous equations, from which the desired sensitivities ($\partial \zeta_i/\partial q_r$) and ($\partial \omega_i/\partial q_r$) can be obtained.

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d. Sensitivity Relations from the Constant Zeta and Omega Loci

In Section 5 it has been shown how the equations of the constant zeta loci $Z(a_k, \zeta_i) = 0$ and the constant omega loci $W(a_k, \omega_i) = 0$ can be obtained in non-parametric form. From these equations we can obtain the relations for the microscopic sensitivity directly:

$$Z(a_k, \zeta_i) = 0 \tag{8-23}$$

$$W(a_k, \omega_i) = 0 \tag{8-24}$$

from which:

$$\partial \zeta_i / \partial a_k = -(\partial Z / \partial a_k)/(\partial Z / \partial \zeta_i) \tag{8-25}$$

providing $\partial Z / \partial \zeta_i \neq 0$

$$\partial \omega_i / \partial a_k = -(\partial W / \partial a_k)/(\partial W / \partial \omega_i) \tag{8-26}$$

providing $\partial W / \partial \omega_i \neq 0$

For an example of sensitivity computed from the non-parametric equations of the constant zeta and omega loci the reader is referred to Section 11 where root sensitivity to parameter variations in a lattice network is discussed in detail.

9. Parameter Plane Analysis and Design of the Symmetrical Parallel-Tee Network

a. Introduction

The symmetrical parallel-tee network shown in Figure 4 has been the subject of several papers (9,10,11). The reason for this interest stems primarily from the fact that it is possible to obtain complex zeros of transmission using the network. These may be placed anywhere in the left half s-plane and as far as 30 degrees from the imaginary axis into the right half s-plane by suitably choosing the network elements. From the transfer function of the network, Equation (9-1), it can be seen that the poles and zeros of transmission are functions of the parameters α and β :

$$\frac{E_o(p)}{E_i(p)} = \frac{\alpha\beta p^3 + 2\beta p^2 + 2\beta p + 1}{\alpha\beta p^3 + (\alpha + 2\beta + 2\alpha\beta)p^2 + (\alpha + 2\beta + 2)p + 1} \qquad (9\text{-}1)$$

where: $p = RCs$

Barker and Rosenstein (12), using root locus techniques, investigated this network. Their analysis, while complete and well done, was complicated considerably by the inherent single parameter capability of the root locus technique. This section approaches the same problem with a two parameter method of analysis and design - the parameter plane.

By employing parameter plane techniques, graphs are prepared for the polynomials which form the numerator and denominator of the transfer function. These graphs, with α as abscissa and β as ordinate, display directly the roots of the polynomial which result for any choice of the parameters α and β . From these parameter planes for the network, the regions of the s-plane in which poles and zeros of transmission are possible

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are immediately evident. Likewise values of the parameters α and β necessary to place either poles or zeros at some desired point within these regions can be directly read from the curves. The parameter plane is therefore a tool of both analysis and design.

Loci of constant bandwidth are drawn upon a parameter plane for this network. These consist of a family of curves along any one of which the frequency at which the response of the network is down N db is constant. For any choice of α and β the frequencies at which the response is down N db can be read directly from these loci. Loci for the conventional -3db bandwidth are presented in this section.

Thus it is possible to select values of the parameters α and β which guarantee suitable s-plane locations for certain poles and zeros while indicating the bandwidth determined by this choice. Conversely α and β may be chosen to guarantee upper and lower -3db frequencies while offering a selection of poles and zeros which may be read from the curves.

The curves which are presented in this section are frequency normalized and hence universal. They may be used to investigate any symmetrical parallel-tee network of the form shown in Figure 4.

The advantage of the parameter plane over other methods which have been used to analyze and design this network are the rapidity and ease with which the capabilities and characteristics can be determined from the curves and with which the parameters may be selected in order to produce desired network behavior.

b. Approach to the Problem

The objective of the analysis to follow is to determine in what manner the poles, zeros, and -3db frequencies of (9-1) change as the parameters α and β are varied. This can be accomplished by preparing three

parameter planes; one each for the poles, zeros and -3db bandwidth of (9-1). Since some of the equations of these parameter planes have already been developed in earlier sections, they will be recalled as appropriate.

c. Poles of the Transfer Function

Since the network shown in Figure 4 is passive RC, all of the poles will lie along the negative real axis in the p-plane. Let σ denote the value of p at which such a pole of (9-1) occurs. We may thus assert that poles occur at those values of σ which satisfy (9-2):

$$\alpha\beta\sigma^3 + (2\beta + 2\alpha\beta + \alpha)\sigma^2 + (2\beta + 2\alpha + 2)\sigma + 1 = 0 \qquad (9\text{-}2)$$

Rewriting (9-2) in the form of a plane algebraic curve in the α - β plane we obtain:

$$(\sigma^3 + 2\sigma^2)\alpha\beta + (2\sigma^2 + 2\sigma)\beta + (\sigma^2 + \sigma)\alpha + (2\sigma + 1) = 0 \qquad (9\text{-}3)$$

Equation (9-3) represents a family of hyperbolas in the α - β plane. One hyperbola occurs for each selected value of σ. Since (9-2) is satisfied by three values of σ for each choice of α and β, it follows that each point in the parameter plane will have three hyperbolic branches passing through it.

The curves represented by (9-3) can be hand calculated for selected values of σ, but since this is a tedious process, a digital computer was used to obtain them. The resulting curves which specify the pole values for any choice of the parameters are shown in Figure 5. One enters this figure with values of α and β and then reads the values of the three poles directly. Alternately one may select values of α and β to produce some selected pole values.

d. <u>Zeros of the Transfer Function</u>

In this instance the problem is much more interesting since complex zeros are possible. The zeros are the roots of:

$$\alpha\beta p^3 + 2\beta p^2 + 2\beta p + 1 = 0 \qquad (9\text{-}4)$$

Since α and β are presumed real it can be asserted that (9-4) has at least one real root. As before let $p = \sigma$ in (9-4) and rewrite the resulting equation as a plane algebraic curve in the $\alpha - \beta$ plane:

$$\sigma^3\alpha\beta + (2\sigma^2 + 2\sigma)\beta + 1 = 0 \qquad (9\text{-}5)$$

Equation (9-5) also represents a family of hyperbolas in the $\alpha - \beta$ plane. These are shown in Figure 6. Note that in some regions it is possible to have three real zeros.

In describing the location of complex zeros of (9-4) the zeta and omega notation familiar to control systems theory will be used:

$$s = -\zeta\omega \pm j\omega\sqrt{1-\zeta^2} \qquad (9\text{-}6)$$

The non-parametric equations for the constant zeta and omega loci which were developed for this network in Section 5 will be employed. The equations of these loci are repeated below. For their derivation the reader is referred to Section 5.

$$Z(\alpha,\beta,\zeta) = -\alpha\beta^3(4\zeta^2 - 1)^3\ [(-32\alpha\zeta^2 + 16)\beta^2 + (64\alpha\zeta^4 + 16\alpha\zeta^2 - 32\zeta^2 - 8\alpha)\beta + (-64\alpha^2\zeta^6 + 48\alpha^2\zeta^4 - 12\alpha^2\zeta^2 + \alpha^2) = 0 \qquad (9\text{-}7)$$

$$W(\alpha,\beta,\omega) = \alpha\beta\omega^3(\alpha^2\beta^2\omega^6 - 2\alpha\beta^2\omega^4 + 2\beta\omega^2 - 1) = 0 \qquad (9\text{-}8)$$

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A digital computer, programmed as described in Section 5, was used to obtain the constant zeta and omega loci which are shown in Figure 6.

e. -3db Constant Bandwidth Curves

One may obtain the desired -3db constant bandwidth curve by employing the procedures which were developed in Section 6. Referring to that section we note that several quantities must be substituted into Equation (6-17) in order to obtain the curve which we desire. In order to facilitate substitution into (6-17) the following quantities are tabulated:

From the transfer function (9-1) using the notation of Equations (6-2) and (6-3):

k	a_k	b_k	c_k	d_k	e_k	f_k	g_k	h_k	
0	0	0	0	1	0	0	0	1	
1	$j\omega_b$	$j2\omega_b$	0	$j2\omega_b$	0	$j2\omega_b$	0	0	
2	$-\omega_b^2$	$-2\omega_b^2$	$-2\omega_b^2$	0	0	$-2\omega_b^2$	0	0	(9-9)
3	0	0	$-j\omega_b^3$	0	0	0	$-j\omega_b^3$	0	

From Equations (6-6) thru (6-9) using values from (9-9) one obtains:

	A	B	C	D	E	F	G	H	
Real	$-\omega_b^2$	$-2\omega_b^2$	$-2\omega_b^2$	1	0	$-2\omega_b^2$	0	1	(9-10)
Imag	ω_b	$2\omega_b$	$-\omega_b^3$	$2\omega_b$	0	$2\omega_b$	$-\omega_b^3$	0	

From Equation (6-16) using the values from (9-10) one obtains:

$$R_1 = 4\omega_b^4 + 4\omega_b^2 + 4\alpha\,\omega_b^4 + 4\alpha^2\omega_b^4 + \alpha^2\omega_b^6$$

$$R_2 = 4\omega_b^4 + 4\omega_b^2 + \alpha^2\omega_b^6 - 4\alpha\,\omega_b^4$$

$$W_1 = 2\alpha^2\omega_b^4 - 4\omega_b^2 \qquad (9\text{-}11)$$

$$W_2 = -4\omega_b^2$$

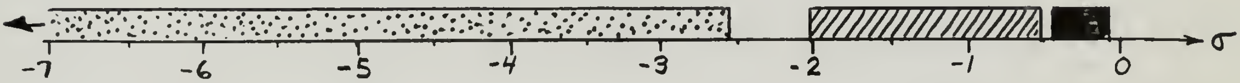
$$V_1 = 1 + \alpha^2\omega_b^4 + \alpha^2\omega_b^2 + 2\alpha\omega_b^2 + 4\omega_b^2$$

$$V_2 = 1$$

The quantities (9-11) are used to digitally compute values of Equation (6-17) corresponding to selected values of ω_b with $M = 1/\sqrt{2}$ as required for the -3db curves. The resulting curves are displayed in Figure 7.

f. Interpretation of the Parameter Planes

The parameter plane for the poles of the transfer function which is shown in Figure 5 reveals that for $0 < \alpha, \beta < 2.5$ the three poles are confined to regions along the negative real axis as shown below:



The parameter plane shown in Figure 6 for the zeros of the transfer function reveals the following information regarding the zeros:

(1) The positive α axis coincides with the $\zeta = -\frac{1}{2}$ locus indicating that zeros may be placed as far as 30 degrees from the imaginary axis into the right half p-plane.

(2) The zeta = 1 locus separates the region of 3 real zeros from the region of 2 complex conjugate zeros and 1 real zero.

(3) The zeta = 0 locus is a straight line with equation $\alpha = 4\beta$. This indicates that if one is interested only in those cases where a pair of conjugate zeros are placed on the $j\omega$ axis, then he may substitute this linear relation between the parameters into the network

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equations thus reducing the problem of analysis and synthesis to one of a single parameter.

(4) The zeta = ½ locus coincides with the omega = 1 locus indicating that an additional degree of freedom exists which may be used to specify other poles or zeros in addition to the specification that a pair of complex conjugate zeros occur at $\zeta = \frac{1}{2}$, $\omega = 1$.

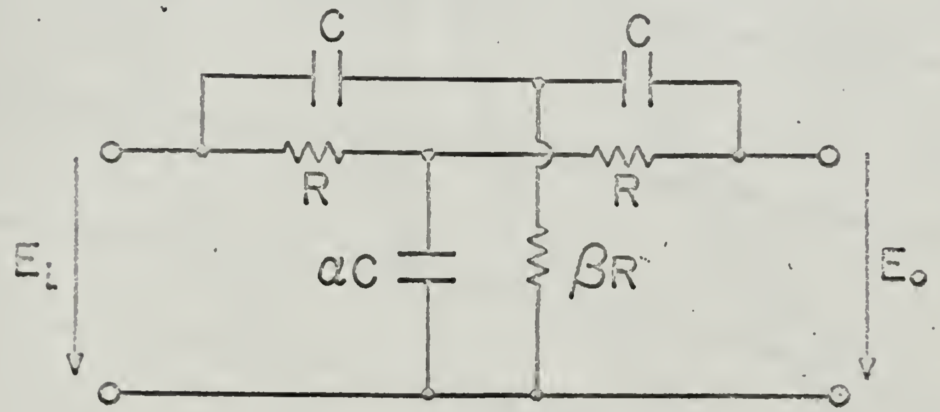


Fig. 4 Symmetrical Parallel-Tee Network

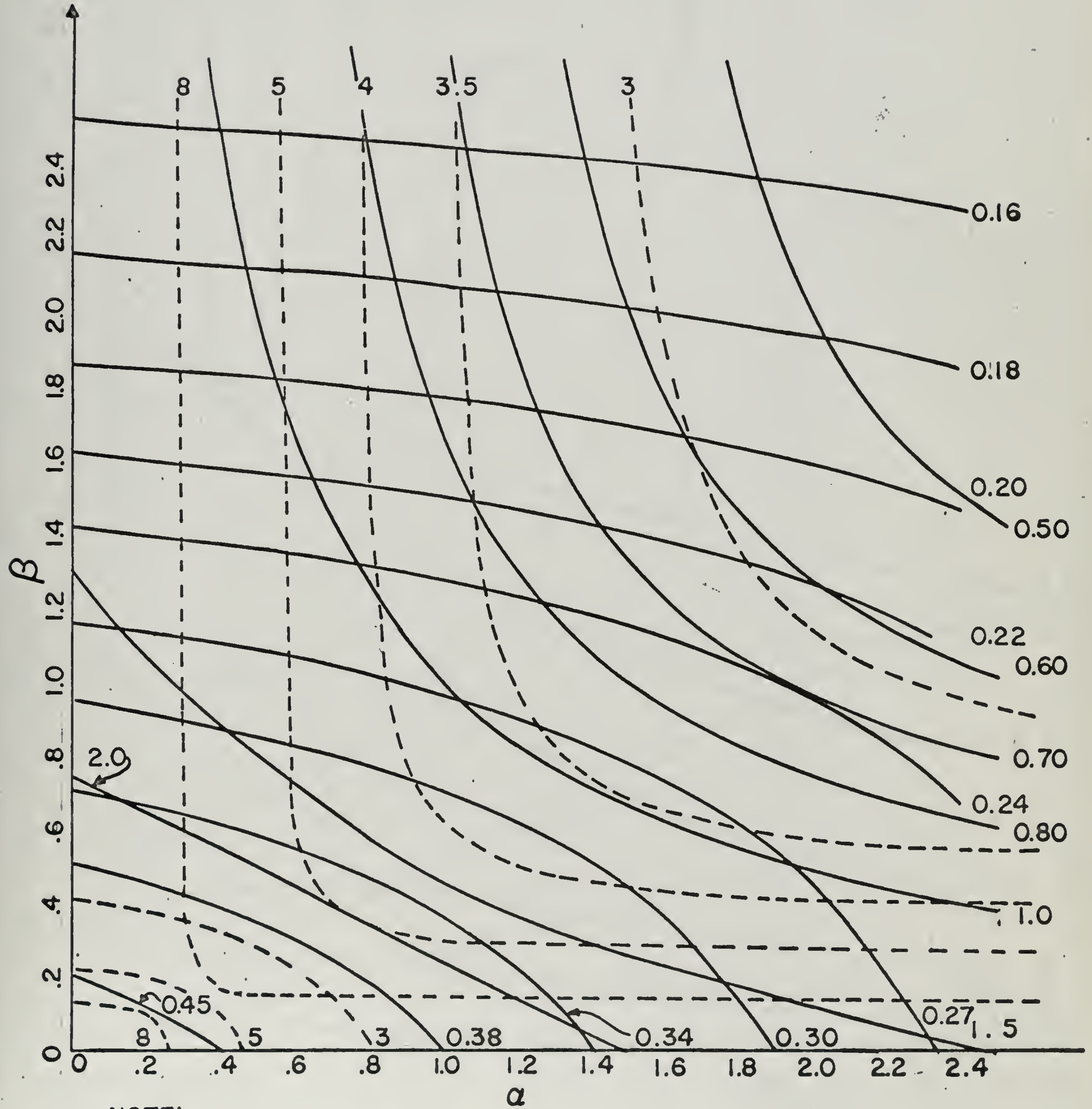


Fig. 5 Poles of the Symmetrical Parallel-Tee Network

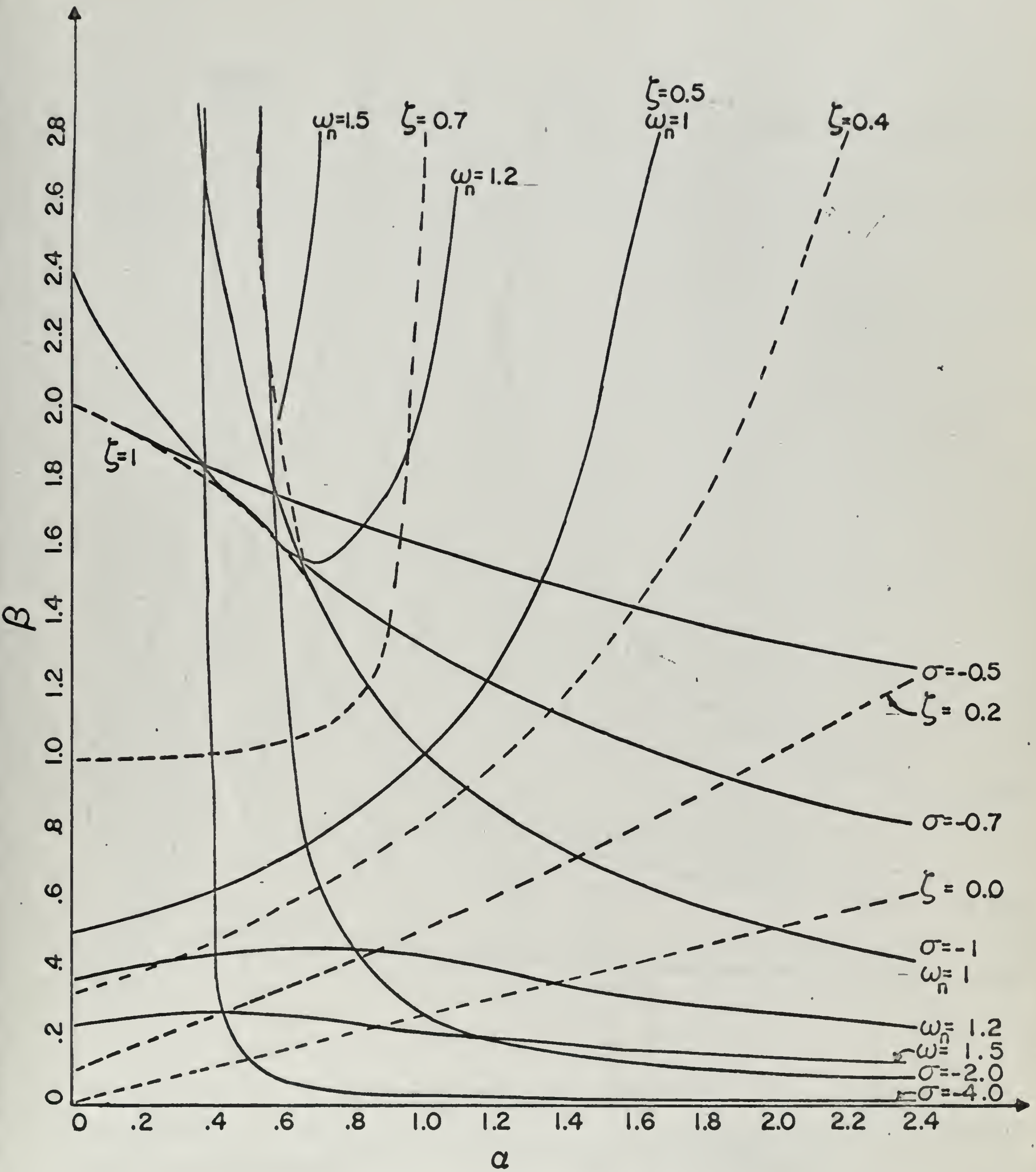
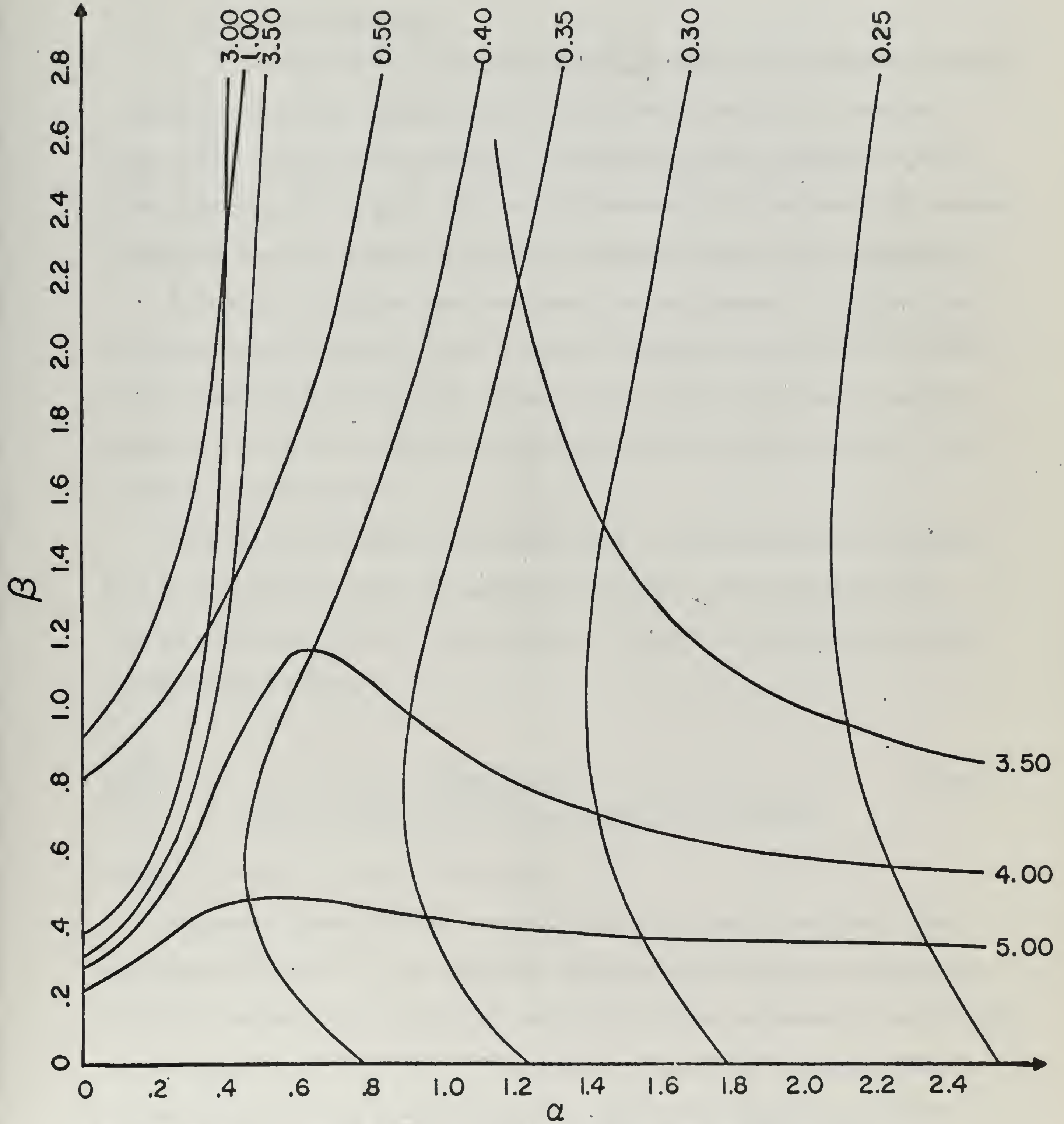


Fig. 6 Zeros of the Symmetrical Parallel-Tee Network



NOTE: CURVE LABELS REFER TO NORMALIZED -3db FREQUENCIES

Fig. 7 -3db Constant Bandwidth Curves for the Symmetrical Parallel-Tee Network

10. Loaded and Null Adjusted Symmetrical Parallel-Tee Network - Parameter Plane Analysis

a. Introduction

The null - adjusted symmetrical parallel-tee network provides a means of obtaining frequency selectivity without the use of inductors. Hence it is a very useful network at low frequency where inductor size becomes unacceptably large as well as in integrated circuits where the current technology does not permit inductors of reasonable size to be integrated.

A number of articles have considered the zero source - infinite load resistance case (9,13,14), and a few have considered the effects of other finite combinations (10,11,12). These latter articles generally consider either the source or resistance to be fixed and the load resistance to be variable or vice-versa.

This section considers the problem from a different point of view and with a two parameter tool of analysis and design - the parameter plane. The two parameter nature of the problem is evident by inspection of Figure 8 and Equation (10-1):

$$\frac{E_o(p)}{E_i(p)} = \frac{\beta\,(p^2 + 1)}{(4\alpha\beta + \alpha + \beta)p^2 + (4\alpha\beta + 4\alpha + 4\beta + 2)p + (\alpha + \beta + 2)} \qquad (10\text{-}1)$$

where: $p = RCs$, $\alpha = R_s/R$, $\beta = R_L/R$

Parameter planes will be obtained for this transfer function. They will show the poles, Q, and frequency response characteristics associated with this network which result for any choice of the parameters α and β within the range $0 < \alpha , \beta < 3$. Alternately α and β may be chosen from the curves to obtain desired characteristics within the capabilities of the

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network.

Loci of constant Q discussed in Section 7 are also obtained. These loci show directly what value of Q results for any selection of parameters α and β . Conditions are specified which permit Q to be related to the bandwidth of the network.

Constant bandwidth curves are obtained as described in Section 6. For any choice of parameters these curves immediately give the upper and lower frequencies where the response of the network is down N db from the response at infinite frequency. Curves are presented for several values of N, thus permitting the frequency response to be sketched for any chosen values of source and load resistance.

b. Poles and Zeros of the Transfer Function

The parameter plane graphically displays the roots of a polynomial whose coefficients are functions of two variable parameters. In the general case these polynomials (which may be the numerator or denominator of some network function) can have complex as well as real roots. In the case of Equation (10-1), however, the following observations are made:

The zeros are fixed at $p^2 = -1$

The poles are all real and lie on the negative real axis in the p-plane since the network is passive RC.

Thus it is necessary to investigate only the poles of (10-1) and this is considerably simplified by the fact that the poles are all real. To obtain the parameter plane for these poles let $p = \sigma$ be a pole of (10-1). Making this substitution into the denominator of (10-1) and rearranging the resulting equation into the form of a conic section in the α - β plane one obtains:

$$(4\sigma^2 + 4\sigma)\alpha\beta + (\sigma^2 + 4\sigma + 1)\alpha + (\sigma^2 + 4\sigma + 1)\beta + (2\sigma + 2) = 0 \tag{10-2}$$

This equation represents a family of hyperbolas; one hyperbola for each value of σ. These are plotted for several values of σ and are shown in Figure 9. One may read the poles, which result for any choice of α and β, directly from the curves.

Example VI

Assume that a parallel-tee notch network has been designed to reject 60 cps. The notch filter has been designed so that R = 2660 ohms and C = 1 microfarad. The network is driven from a source resistance of 5000 ohms and drives a load resistance of 2500 ohms. It is required that the poles of the transfer function be obtained. From Figure 8 one obtains:

$$\alpha = \frac{5000}{2660} = 1.88$$

$$\beta = \frac{2500}{2660} = 0.94$$

From Figure 9 the normalized poles are read:

$$\sigma_1 = -1.78$$

$$\sigma_2 = -0.27$$

Which, when un-normalized become:

$$\sigma_1' = \frac{-1.78}{2.66 \times 10^{-3}} = -670$$

$$\sigma_2' = \frac{-0.27}{2.66 \times 10^{-3}} = -102$$

The network transfer function is thus:

$$\frac{E_o(s)}{E_i(s)} = \frac{\beta}{4\alpha\beta + \alpha + \beta} \left[\frac{s^2 + \omega_o^2}{(s + \sigma_1')\,(s + \sigma_2')} \right]$$

$$= 0.95 \left[\frac{s^2 + 377^2}{(s+670)\;(s+102)} \right]$$

The curves of Figure 9 can be used to place one of the poles at a desired location while maintaining some control over the location of the other pole. This is accomplished by changing α, β, or R. Changing R and assuming that C is also changed appropriately to keep the zeros fixed changes both α and β.

<u>Example VII</u>

Suppose that in the previous example one is able to change the source resistance and desires to do so in such a manner as to cause the pole at s = -670 to move to s = -750. Thus the new normalized pole location becomes: $p_1 = -2.00$. Since the load resistance is fixed at 2500 ohms, β remains 0.94. From Figure 9 note that the $\sigma = -2.00$ locus intersects the $\beta = 0.94$ coordinate at $\alpha = 1.07$, and that the other pole is thus located at $p_2 = -0.335$. When un-normalized these pole locations become:

$$s_1 = -750$$

$$s_2 = -126$$

The new transfer function becomes:

$$\frac{E_o(s)}{E_i(s)} = 0.156 \left[\frac{s^2 + 377^2}{(s + 750)(s + 126)} \right]$$

The new source resistance required becomes:

$$R_s = \alpha R = 1.07(2660) = 2840 \text{ ohms}$$

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c. Constant Bandwidth Curves

By making the substitution $p = j\omega$ as discussed in Section 6, the real frequency transfer function is obtained for the loaded and null adjusted symmetrical parallel-tee network:

$$\frac{E_o(\omega)}{E_i(\omega)} = K(\alpha, \beta)\,[\,G(\alpha, \beta, \omega)] \tag{10-3}$$

Where:

$$K(\alpha, \beta) = \frac{\beta}{4\alpha\beta + \alpha + \beta} \tag{10-4}$$

$$G(\alpha, \beta, \omega) = \frac{(4\alpha\beta + \alpha + \beta)(1-\omega^2)}{[(\alpha+\beta+2)-(4\alpha\beta+\alpha+\beta)\,\omega^2] + j[\,(4\alpha\beta+4\alpha+4\beta+2)]} \tag{10-5}$$

By inspection $G(\alpha, \beta, \omega) \rightarrow 1.0$ as $\omega \rightarrow \infty$. At any selected frequency ω_b, and attenuation level M at that frequency, the relation

$$G(\alpha, \beta, \omega_b) = M \tag{10-6}$$

defines a constant bandwidth curve in the α - β plane. This curve specifies the source-load resistance relation requires for (10-6) to be satisfied. The attenuation level M can be thought of as the attenuation from the infinite frequency value of $E_o(\infty)/E_i(\infty)$, which is $K(\alpha, \beta)$:

$$K(\alpha, \beta) = \frac{E_o(\infty)}{E_i(\infty)} \tag{10-7}$$

In order to obtain the equation of the constant bandwidth curves Equation (10-5) is substituted into Equation (10-6) and both sides of the resulting equation is squared. This squared equation can be solved for β as a function of α, ω_b, and M:

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$$\beta = \frac{-(M^2W_1-W_2) \pm [(M^2W_1-W_2)^2-4(M^2R_1-R_2)(M^2V_1-V_2)]^{\frac{1}{2}}}{2(M^2R_1-R_2)} \qquad (10\text{-}8)$$

Where in the notation of Section 6:

$$R_1 = \omega^4 + 14\omega^2 + 1 + 4\alpha(2\omega^4 + 6\omega^2) + 16\alpha^2(\omega^4 + \omega^2)$$

$$R_2 = \omega^4 - 2\omega^2 + 1 + 4\alpha(2\omega^4-4\omega^2+2) + 16\alpha^2(\omega^4-2\omega^2 + 1)$$

$$W_1 = 12\omega^2 + 4 + 2\alpha(\omega^4 + 14\omega^2 + 1) + 4\alpha^2(2\omega^4 + 6\omega^2)$$

$$W_2 = 2\alpha(\omega^4-2\omega^2 + 1) + 8\alpha^2(\omega^4-2\omega^2 + 1)$$

$$V_1 = 4 + 4\omega^2 + 4\alpha(3\omega^2 + 1) + \alpha^2(\omega^4 + 14\omega^2 + 1)$$

$$V_2 = \alpha^2(\omega^4-2\omega^2 + 1)$$

Assisted by a digital computer, Equation (10-8) was used to obtain the constant bandwidth curves for values of M corresponding to -3db, -6db, and -20db. These curves are shown in Figures 10, 11, and 12 respectively. These curves may be used to sketch the frequency response as indicated in the following example.

Example VIII

For the conditions specified in Example VI, namely $\alpha = 1.88$ and $\beta = 0.94$, the curves of Figures 10, 11, 12 are used to obtain:

Db down from infinite frequency gain	ω_{b_1}	ω_{b_2}
-3	0.50	2.65
-6	0.61	1.84
-20	0.901	1.115

The infinite frequency gain is:

$$K(1.88,\ 0.94) = 0.095$$

The zero frequency (DC) is:

$$\frac{E_o(0)}{E_i(0)} = \frac{\beta}{\alpha + \beta + 2} = 0.195$$

The attenuation at $\omega = 1.0$ is infinite since the network of Example VI was designed for this condition.

The above information may be used to sketch the frequency response of the network.

d. Constant Q Loci

As pointed out in Section 7, the Q of a second order network may be defined as:

$$A = \frac{\text{geometric mean of } 0^{th} \text{ and } 2^{nd} \text{ degree terms}}{1^{st} \text{ degree term}}$$

where these terms refer to coefficients in the denominator or the network transfer function. For the loaded and null-adjusted symmetrical parallel-tee network, Q becomes:

$$Q = \frac{[(4\alpha\beta + \alpha + \beta)(\alpha + \beta + 2)]^{\frac{1}{2}}}{4\alpha\beta + 4\alpha + 4\beta + 2} \qquad (10\text{-}9)$$

Squaring both sides of Equation (10-9) and rearranging the resulting equation into the form of a quadratic in β obtains:

$$\begin{aligned}
&[16Q^2\alpha^2 + (32Q^2-4)\alpha + (16Q^2-1)]\beta^2 + \\
&\quad [(32Q^2-4)\alpha^2 + (48Q^2-10)\alpha + (16Q^2-2)]\beta + \\
&\quad\quad [(16Q^2-1)\alpha^2 + (16Q^2-2)\alpha + 4Q^2] = 0
\end{aligned} \qquad (10\text{-}10)$$

Equation (10-10) was used to obtain the constant Q loci displayed in Figure 13.

Q defined in this manner is a mathematical parameter of the network as discussed in Section 7. It is not in general directly related to the frequency response, however, if the infinite frequency response is made equal to the DC response as explained later, then the following relationship holds:

$$Q = \frac{f_0}{f_2 - f_1} \tag{10-11}$$

Where: f_0 is the notch frequency

f_1 is the lower -3db frequency

f_2 is the upper -3db frequency

It has been shown (10) that the null adjusted symmetrical parallel-tee network under the conditions $R_s = 0$, $R_L =$ has a Q equal to 0.25. Figure 13 shows that Q may be made larger than 0.25 by adding source resistance. The maximum attainable Q is $.353^{+}$ and occurs at $R_s = R_L = 0.707R$.

e. <u>Locus of Symmetrical Frequency Response</u>

Also displayed on Figure 13 is the locus of symmetrical frequency response. This locus defines the source - load condition required for the DC response to equal the infinite frequency response. Cowles (10) has shown that this requires $R_s R_L = R^2/2$ or equivalently $\alpha\beta = \frac{1}{2}$. Figure 13 shows that this locus also passes thru the point of maximum Q. Thus a loaded and null adjusted symmetrical parallel-tee network whose source and load resistances are chosen so as to maximize Q will also have a symmetrical frequency response. As pointed out above in sub-section d, the network under these conditions also is the most selective by virtue of its optimum Q.

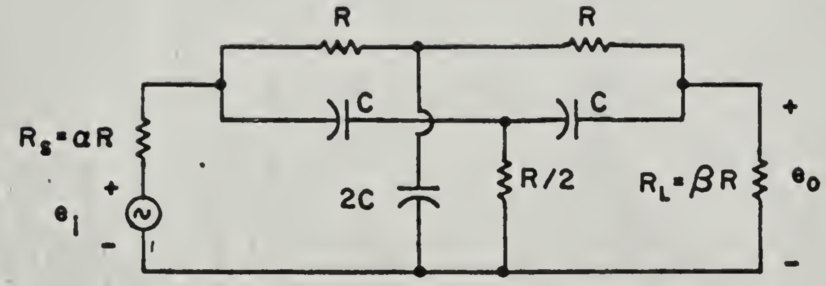

Fig. 8 Loaded and Null Adjusted Symmetrical Parallel-Tee Network

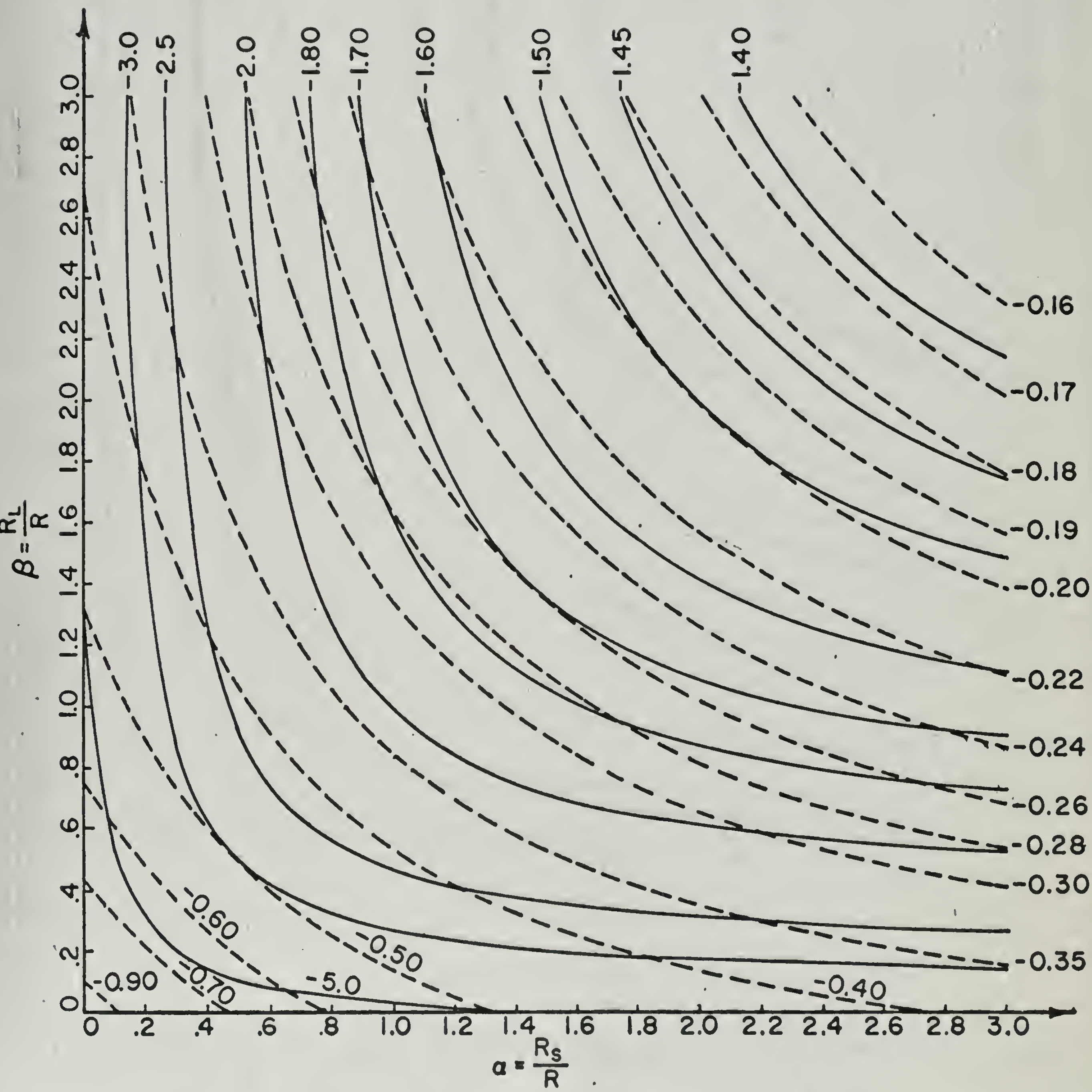

Fig. 9 Poles of the Loaded and Null Adjusted Symmetrical Parallel-Tee Network

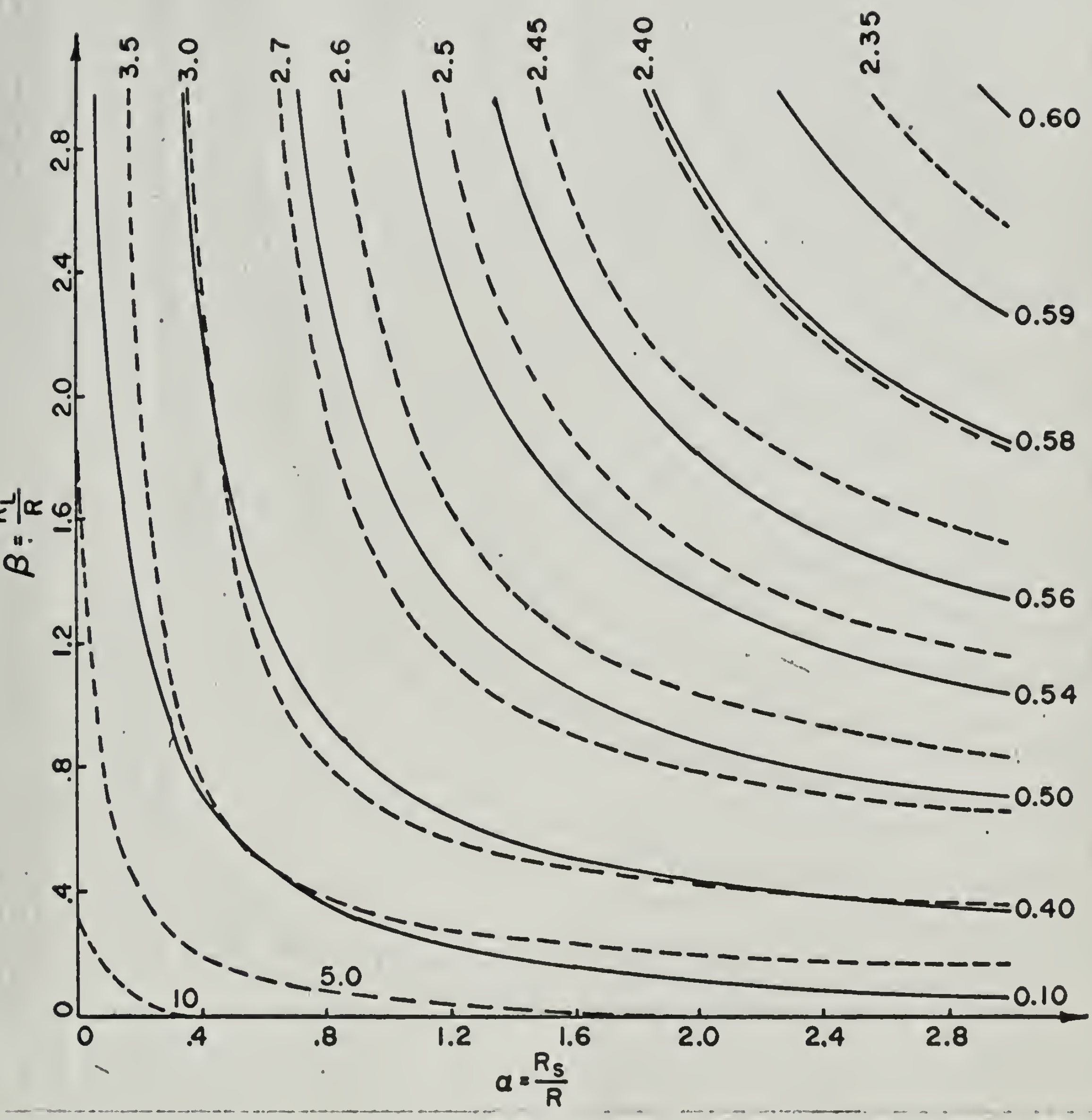

Fig. 10 -3db Constant Bandwidth Curves for the Loaded and Null Adjusted Symmetrical Parallel-Tee Network

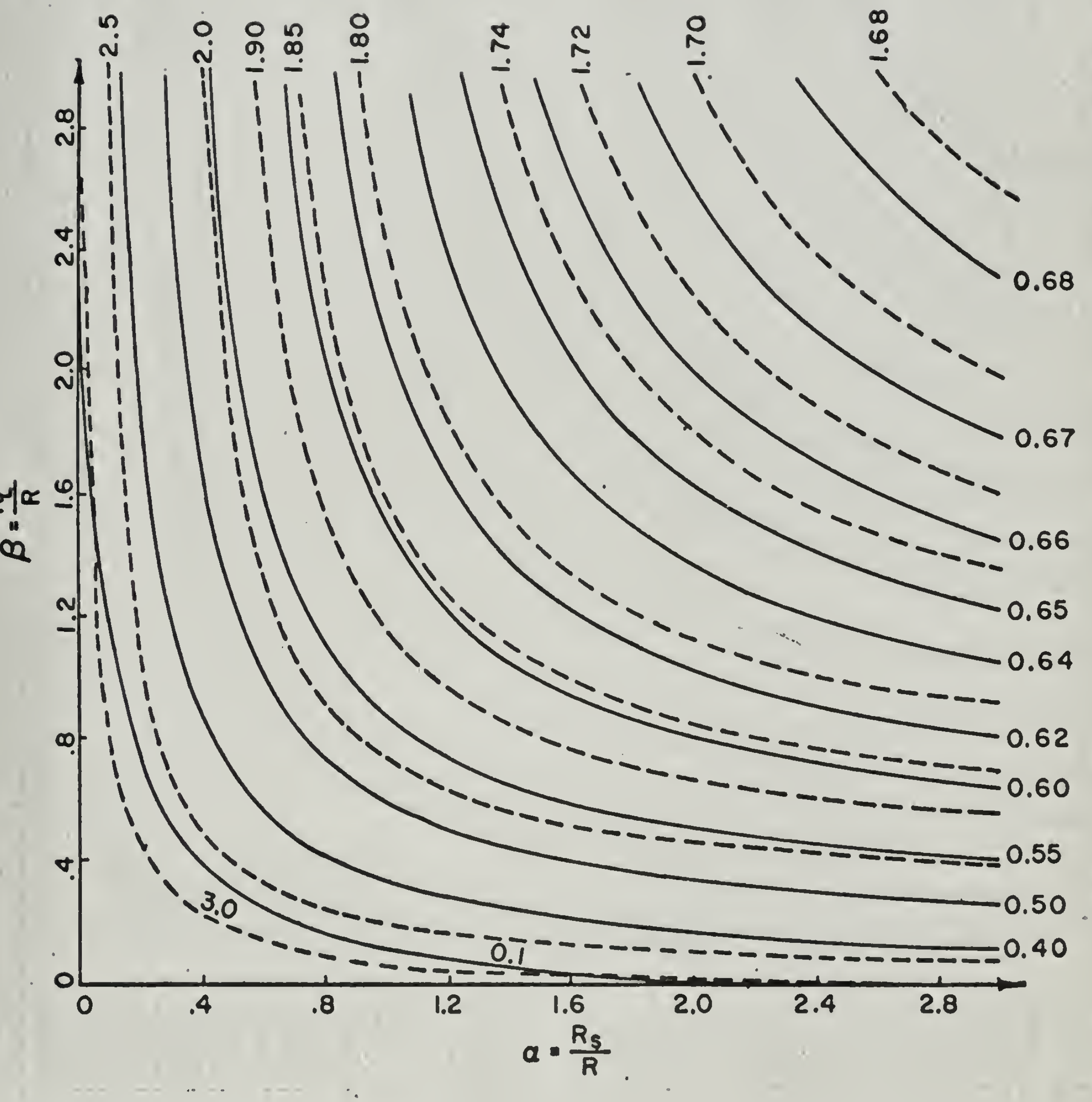

Fig. 11 -6db Constant Bandwidth Curves for the Loaded and Null Adjusted Symmetrical Parallel-Tee Network

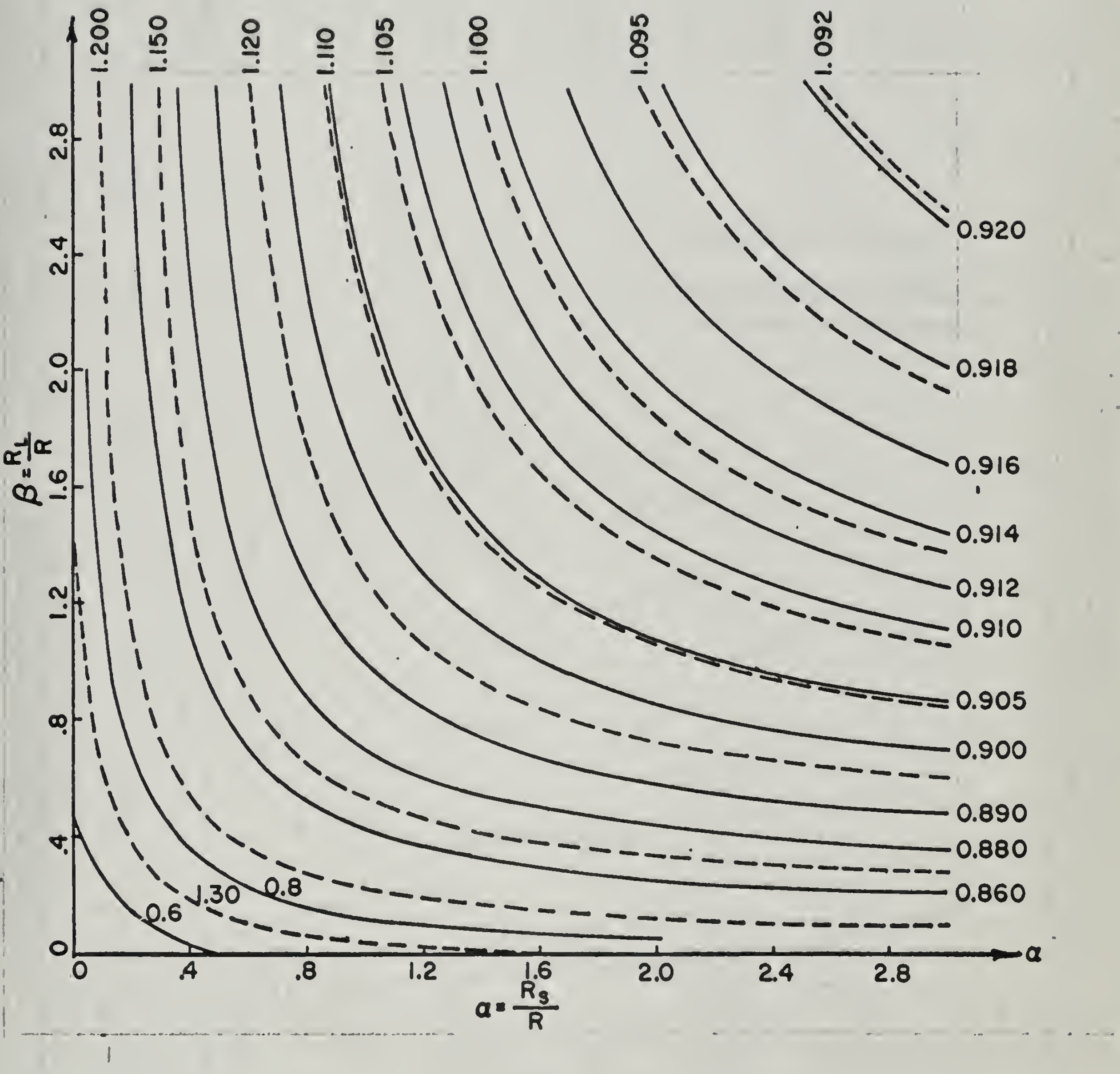

Fig. 12 -20 db Constant Bandwidth Curves for the Loaded and Null Adjusted Symmetrical Parallel-Tee Network

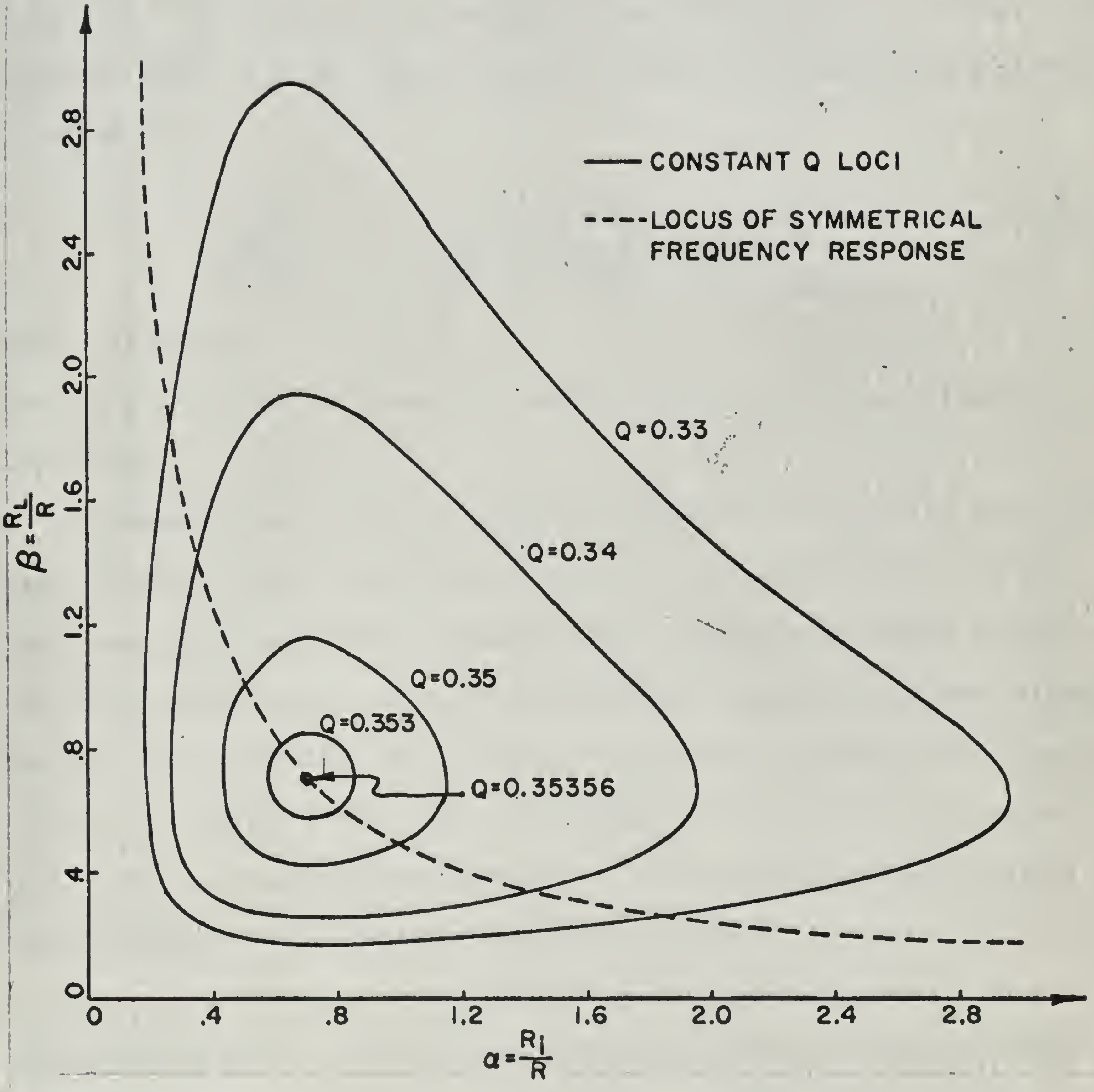

Fig. 13 Constant Q Loci and Locus of Symmetrical Frequency Response for the Loaded and Null Adjusted Symmetrical Parallel-Tee Network

11. RC Lattice Network - Complex Zero Synthesis and Sensitivity Analysis by Parameter Plane Techniques

a. Introduction

The RC lattice network which is investigated is shown in Figure 14. The parameters α and β are real and variable in the arbitrarily selected range $0 < \alpha, \beta < 2.5$. The transfer function of this network is given by:

$$\frac{E_o(p)}{E_i(p)} = \frac{\alpha\beta\ p^2 + (\alpha\beta - \beta + 1)p + 1}{\alpha\beta\ p^2 + (\alpha\beta + \beta + 1)p + 1} \tag{11-1}$$

where: $p = RCs$

Frequency normalization makes the results which are obtained universally applicable.

Parameter planes which are obtained graphically display the poles and zeros of (11-1) which occur for any choice of parameters α and β within the range specified. These parameter planes show in what manner the poles and zeros change as the parameters are varied. Using the parameter planes one may select parameter values which are required in order to place poles or zeros at desired p-plane locations within the capabilities of the network. The analysis performed shows that it is possible to readily place zeros anywhere in the complex region of the p-plane.

Macroscopic root sensitivity can be determined by inspection of the parameter planes. Equations are obtained as described in Section 8 which permit microscopic root sensitivity to be computed for selected root locations.

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$$[illegible] \qquad (3.11)$$

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b. Poles and Zeros of the Transfer Function

The poles of the transfer function are given by the roots of:

$$\alpha\beta p^2 + (\alpha\beta + \beta + 1)p + 1 = 0 \qquad (11\text{-}2)$$

Since the network of Figure 14 is passive RC, the roots of (11-2) all lie on the negative real axis in the p-plane. Let $p = \sigma$ be such a root of (11-2). Making this substitution into (11-2) and rearranging the resulting equation into the form of a conic section in the α - β plane one obtains:

$$(\sigma^2 + \sigma)\alpha\beta + \sigma\beta + (\sigma + 1) = 0 \qquad (11\text{-}3)$$

Equation (11-3) represents a family of hyperbolas in the α - β plane; one hyperbola results for each value of σ. This equation is solved for β as a function of α and σ and used to graph the curves displayed in Figure 15:

$$\beta = \frac{-(\sigma + 1)}{(\sigma^2 + \sigma)\alpha + \sigma} \qquad (11\text{-}4)$$

The zeros of the transfer function are the roots of:

$$\alpha\beta p^2 + (\alpha\beta + 1 - \beta)p + 1 = 0 \qquad (11\text{-}5)$$

Following the procedures of Section 5 the constant zeta and omega loci are obtained in non-parametric form:

(1) Constant Zeta Loci

$$f(\omega) = 2\zeta\alpha\beta\omega^2 + (\beta - \alpha\beta - 1)\omega + 0 = 0 \qquad (11\text{-}6)$$

$$g(\omega) = 0\omega^2 + (\beta - \alpha\beta - 1)\omega + 2\zeta = 0 \qquad (11\text{-}7)$$

From which:

$$Z(\alpha,\beta,\zeta) = \begin{vmatrix} 2\zeta\alpha\beta & (\beta-\alpha\beta-1) & 0 & 0 \\ 0 & 2\zeta\alpha\beta & (\beta-\alpha\beta-1) & 0 \\ 0 & (\beta-\alpha\beta-1) & 2\zeta & 0 \\ 0 & 0 & (\beta-\alpha\beta-1) & 2\zeta \end{vmatrix} = 0 \qquad (11\text{-}8)$$

Expanding (11-8) one obtains:

$$Z(\alpha,\beta,\zeta) = -4\zeta^2\alpha\beta \;[(\alpha^2-2\alpha+1)\beta^2 + (-4\zeta^2\alpha+2\alpha-2)\beta + 1] = 0 \quad (11\text{-}9)$$

Thus:

$$\beta = \frac{(4\zeta^2\alpha-2\alpha+2) \pm \sqrt{(-4\zeta^2\alpha+2\alpha-2)^2 - 4(\alpha^2-2\alpha+1)}}{2(\alpha^2-2\alpha+1)} \quad (11\text{-}10)$$

gives the constant zeta loci everywhere except where $4\zeta^2\alpha\beta = 0$. Equation (11-10) was used to obtain the constant zeta loci shown in Figure 16.

(2) Constant Omega Loci

Using Theorem II, Appendix II, for the cases j = 0 and j = 1, one obtains:

$$\begin{bmatrix} -(\beta-\alpha\beta-1)\omega & +\alpha\beta\omega^2 \\ (\alpha\beta\omega^2-1) & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \underline{0} \quad (11\text{-}11)$$

From which:

$$W(\alpha,\beta,\omega) = \alpha\beta\omega^2(\alpha\beta\omega^2-1) = 0 \quad (11\text{-}12)$$

Thus $\alpha\beta = 1/\omega^2$ gives the constant omega loci everywhere except where $\alpha\beta\omega^2 = 0$. Equation (11-12) was used to obtain the constant omega loci shown in Figure 16.

c. Root Sensitivity

(1) Macroscopic Sensitivity

Root variations due to large parameter variations are readily observed by inspecting Figures 15 and 16. In the case of the sensitivity of the zeros of the transfer function, one can see that zeros with $0.3 < \zeta < 0.7$, and $0.5 < \omega < 1.0$ are relatively insensitive to parameter variations. Given a specification on permissible zeta and omega variation, one can determine, practically by inspection, the required α and β tolerances

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which will meet the specification.

(2) Microscopic Sensitivity

Zero sensitivity due to infinitesmal variations in the parameters may be readily computed from formulas derived from the non-parametric equations of the constant zeta and omega loci as discussed in Section 8. Substituting derivatives obtained from (11-9) and (11-12) into (8-25) and (8-26) one obtains:

$$\frac{\partial \omega_i}{\partial \alpha} = \frac{-\omega}{2\alpha} \tag{11-13}$$

$$\frac{\partial \omega_i}{\partial \beta} = \frac{-\omega}{2\beta} \tag{11-14}$$

where: $\alpha(4\alpha\beta\omega^2 + 2) \neq 0$

$$\frac{\partial \zeta_i}{\partial \alpha} = \frac{-\zeta(8\zeta^2\alpha\beta - 3\alpha^2\beta^2 + 4\alpha\beta^2 - 4\alpha\beta - \beta^2 + 2\beta - 1)}{2\alpha(8\zeta^2\alpha\beta - \alpha^2\beta^2 + 2\alpha\beta^2 - \beta^2 - 2\alpha\beta + 2\beta - 1)} \tag{11-15}$$

$$\frac{\partial \zeta_i}{\partial \beta} = \frac{-\zeta(8\zeta^2\alpha\beta - 3\alpha^2\beta^2 + 6\alpha\beta^2 - 3\beta^2 - 4\alpha\beta + 4\beta - 1)}{2\beta(8\zeta^2\alpha\beta - \alpha^2\beta^2 + 2\alpha\beta^2 - \beta^2 - 2\alpha\beta + 2\beta - 1)} \tag{11-16}$$

where: $\beta\ (8\zeta^2\alpha\beta - \alpha^2\beta^2 + 2\alpha\beta^2 - \beta^2 - 2\alpha\beta + 2\beta - 1) \neq 0$

d. Interpretation of the Curves

Inspection of the parameter plane for the zeros of the transfer function shows that it is possible to place a pair of complex conjugate zeros anywhere in the s-plane by appropriately choosing the parameter values α, β and the frequency normalizing factor RC.

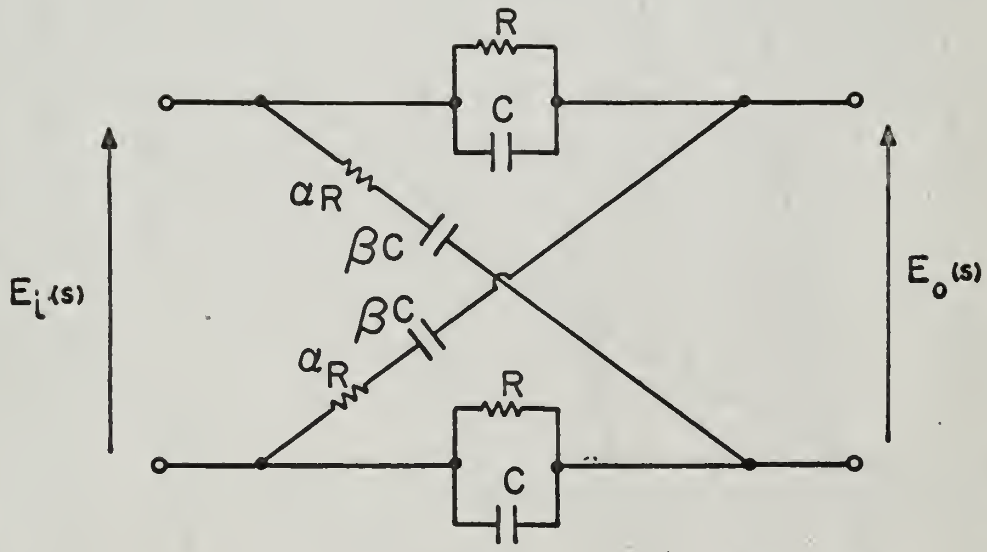


Fig. 14 RC Lattice Network with Variable Parameters

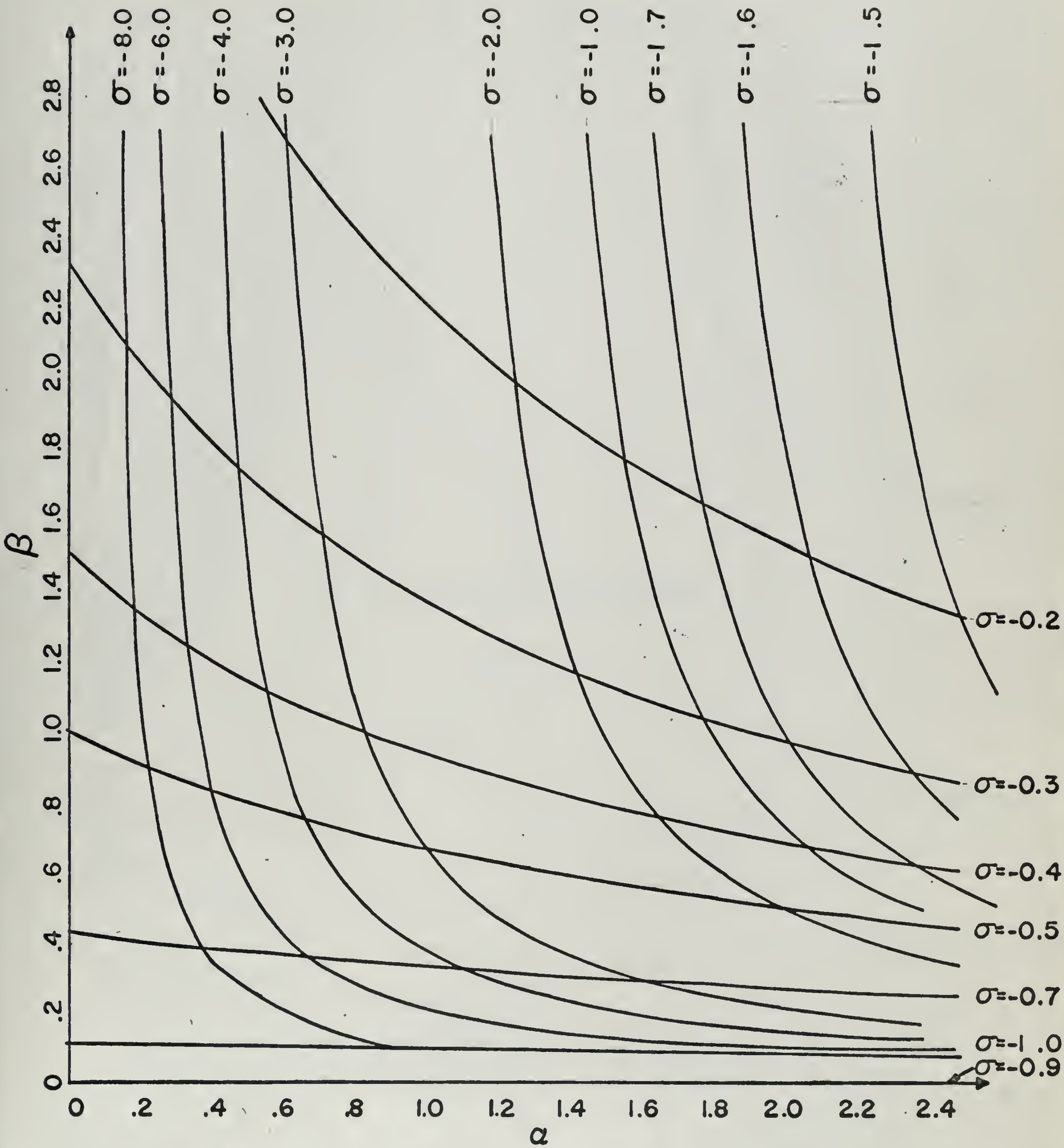


Fig. 15 Poles of the Variable Parameter Lattice Network

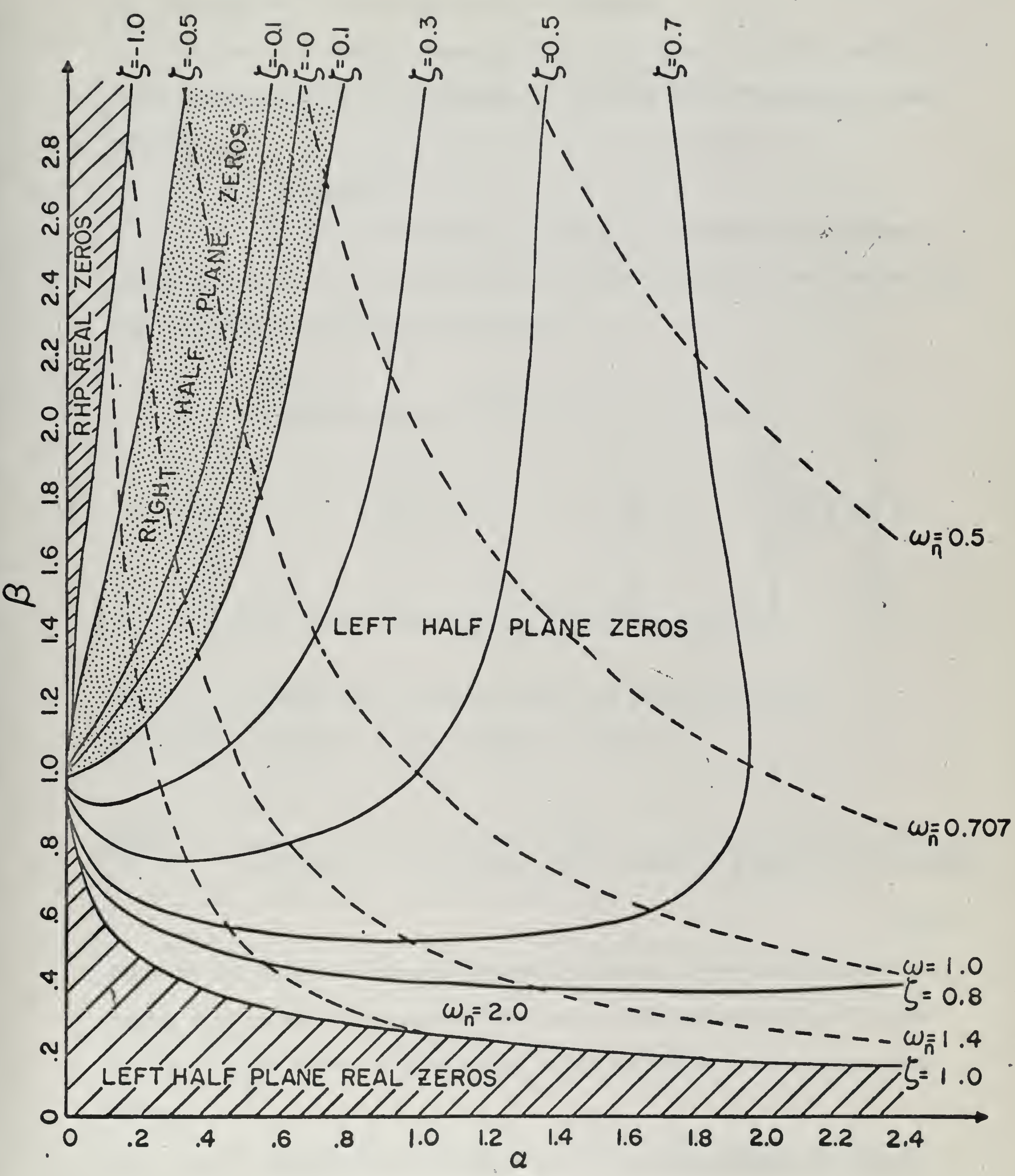


Fig. 16 Zeros of the Variable Parameter Lattice Network

12. Application of the Method to other Problems

This section contains examples of several networks whose transfer functions are examined to determine the applicability of parameter plane methods.

a. Pi Network

The pi network shown in Figure 17 is useful for matching a resistive source to a resistive load at a fixed frequency. The source and load resistances are chosen as variable.

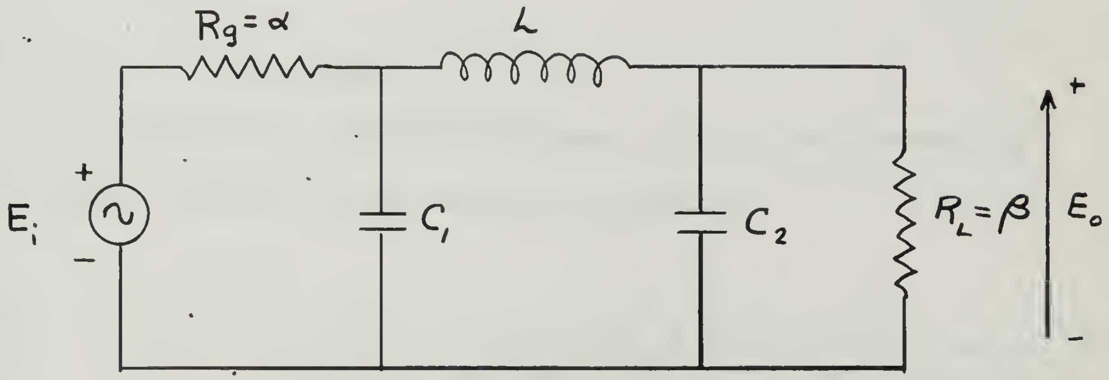


Fig. 17 Pi Network with Variable Source and Load Resistances

The transfer function of this network is given by:

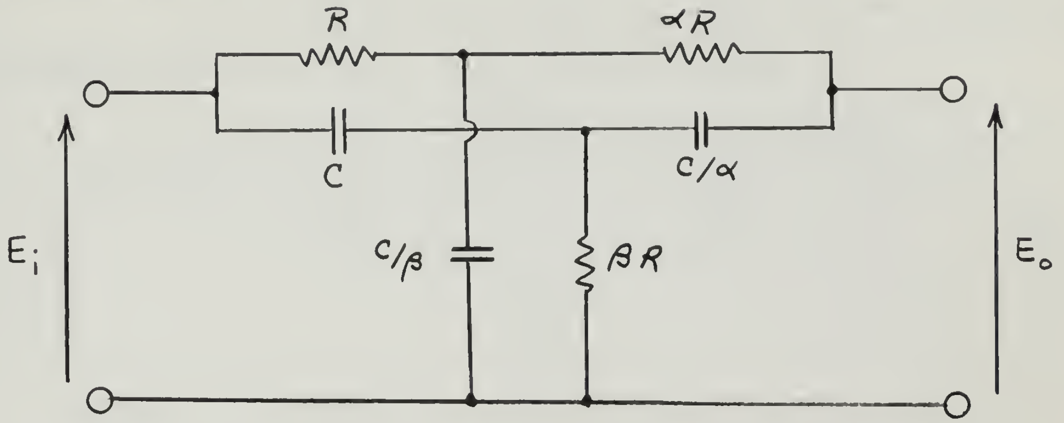
$$\frac{E_o(s)}{E_i(s)} = \frac{\beta}{\alpha/\beta\ C_1C_2s^3 + (\alpha C_1L + \beta C_2L)s^2 + (\alpha/\beta\ C_1 + \alpha/\beta\ C_2 + L)s + (\alpha+\beta)} \tag{12-1}$$

Equation (12-1) shows that the parameters appear in the coefficients in a manner that permits a parameter plane to be constructed which will show how the network poles are determined by the values of the variable parameters.

b. Unsymmetrical Parallel-T Network with Variable Parameters

The network shown in Figure 18 is of particular interest here

because of the way that the parameters appear in the coefficients of the transfer function.



p = RCs

Fig. 18 Unsymmetrical Parallel-T Network with Variable Parameters

The transfer function of this network is given by:

$$\frac{E_o(s)}{E_i(s)} = \frac{\alpha\beta\ p^3 + (\alpha\beta^2 + \beta^2)p^2 + (\alpha\beta^2 + \beta^2)p + \alpha\beta}{\alpha\beta\ p^3 + (\alpha\beta^2 + \beta^2 + \alpha\beta + \alpha + \beta)p^2 + (\alpha\beta^2 + \beta^2 + \alpha\beta + \alpha + \beta)p + 1} \tag{12-2}$$

First inspection of (12-2) indicates that the β^2 terms will complicate the application of parameter plane techniques. The following is noted however:

(1) The network is passive RC, therefore the poles are all real and lie on the negative real axis in the p-plane. Hence $p = \sigma$ may be substituted into the denominator of (12-2). This denominator is then equated to zero and the resulting equation solved by the quadratic formula for β as a function of α and σ . Thus the parameter plane for the poles may be constructed.

(2) The quantity β may be factored from the numerator of (12-2). The resulting polynomial is of the proper form to be analyzed by parameter plane methods.

c. Bridged-T Network with Variable Parameters

The bridged-t network shown in Figure 19 can be used to generate complex conjugate zeros in the left half s-plane.

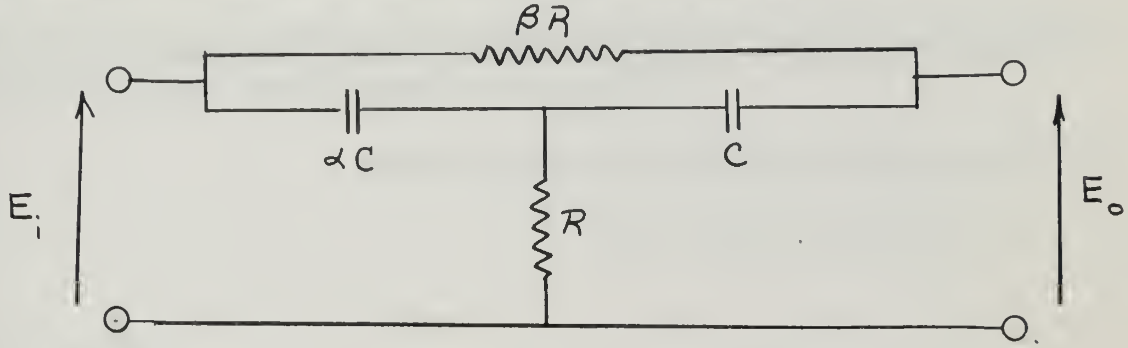


Fig. 19 Bridged-T Network with Variable Parameters

The transfer function of this network is:

$$\frac{E_o(p)}{E_i(p)} = \frac{\alpha\beta\ s^2 + (\alpha + 1)s + 1}{\alpha\beta\ s^2 + (\alpha + \beta + 1)s + 1} \cdot \qquad (12\text{-}3)$$

where p = RCs

The form of Equation (12-3) readily permits parameter plane analysis and design.

d. Bridged Twin-T Network with Variable Parameters

The network shown in Figure 20 is an RC Bridged Twin-T Network with variable bridging and shunting elements.

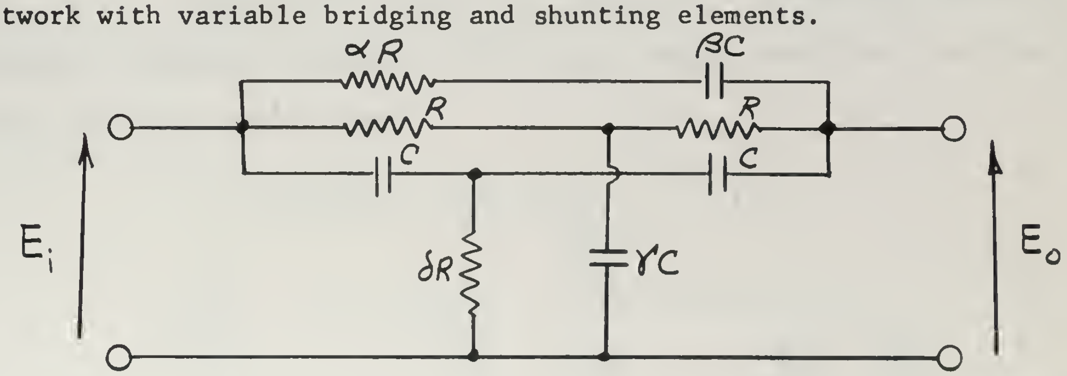


Fig. 20 Bridged Twin-T Network with Variable Parameters

The transfer function of this network is given by (12-4):

$$\frac{E_o(p)}{E_i(p)} = \frac{(\alpha\beta\gamma\delta)p^4 + (2\alpha\beta\delta + 2\beta\gamma\delta + \gamma\delta)p^3 + (4\beta\delta + 2\alpha\beta\delta + \beta\gamma + 2\delta)p^2 + (2\beta + \alpha\beta + 2\delta)p + 1}{(\alpha\beta\gamma\delta)p^4 + (2\alpha\beta\gamma\delta + 2\beta\gamma\delta + 2\alpha\beta\delta + \alpha\beta\gamma + \gamma\delta)p^3 + (2\alpha\beta\delta + \alpha\beta\gamma + 2\alpha\beta + 4\beta\delta + 2\gamma\delta + \beta\gamma + \gamma + 2\delta)p^2 + (\alpha\beta + 2\beta + 2\delta + 2)p + 1} \quad (12\text{-}4)$$

Where p = RCs

Inspection of (12-4) reveals that one may construct several parameter planes depending on which pair of the quantities α, β, γ, δ, are considered variable and which pair are considered fixed. The following possibilities are available:

Variable Parameters	Fixed Parameters
α, β	γ, δ
γ, δ	α, β
α, γ	β, δ
β, δ	α, γ
α, δ	β, γ
β, γ	α, δ

e. Capacitive Divider Matching Network

The network shown in Figure 21 is frequently used for frequency selective coupling between high output resistance and low input resistance amplifier stages.

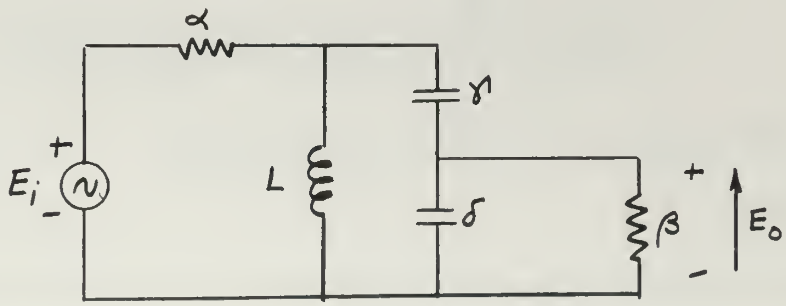


Fig. 21 Capacitive Divider Matching Network

The transfer function of this network is:

$$\frac{E_o(s)}{E_i(s)} = \frac{\beta\gamma\ Ls^2}{(\beta\gamma\delta\ L^2)s^4+(\beta\gamma\delta L^2+\gamma L^2)s^3+(\alpha\beta\gamma\ L-L^2\gamma+\beta\delta\ L+\beta\gamma\ L)s^2+(\alpha\gamma\ L+\alpha\beta\delta+\alpha\beta\gamma+L)s+\alpha} \qquad (12\text{-}5)$$

Equation (12-5) shows that it is possible to construct parameter planes with the capacitances γ and δ variable and the source and load resistances α and β fixed or vice versa.

f. Two Section Resistively Loaded LC Low Pass Filter

The LC low pass filter shown in Figure 22 has variable source and load resistances.

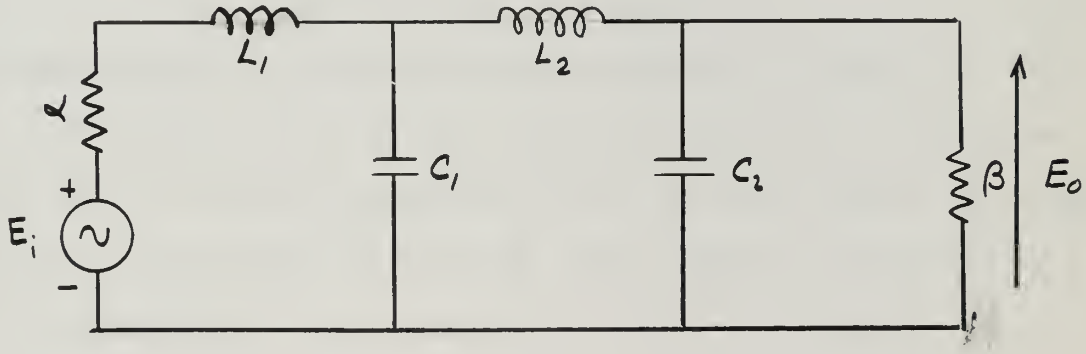


Fig. 22 Two Section Resistively Loaded LC Low Pass Filter

The transfer function of this network is:

$$\frac{E_i(s)}{E_o(s)} = \frac{\beta}{(\beta L_1L_2C_1C_2)s^4+(\alpha\beta\ C_1C_2L_2+L_1L_2C_1)s^3+(\beta L_1C_2+\beta L_1C_1+\beta L_2C_2)s^2+(\alpha\beta\ C_2+\alpha\beta\ C_1+L_1+L_2)s+(\alpha+\beta)} \qquad (12\text{-}6)$$

The form of this function permits application of parameter plane techniques.

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g. <u>Butterworth Filter Pole Sensitivity to Coefficient Variations</u>

Parameter plane techniques can be used to investigate the effects upon pole locations of Butterworth Filters due to variations in the coefficients of the transfer function caused by component tolerances, aging, etc. As an example, consider the transfer function of a normalized 4th order Butterworth filter:

$$\frac{E_o(p)}{E_i(p)} = \frac{1}{p^4 + 2.613p^3 + 3.414p^2 + 2.613p + 1} \tag{12-7}$$

Assume that the last two coefficients in the denominator of (12-7) are variable. Denote these variable coefficients as α and β and equate the denominator of (12-7) to zero. This results in the polynomial equation:

$$p^4 + 2.613p^3 + 3.414p^2 + \alpha p + \beta = 0 \tag{12-8}$$

The parameter plane for this polynomial is displayed in Figure 23. When $\alpha = 2.613$ and $\beta = 1.0$ as is the normal case, the poles of (12-7) are equally spaced on the unit semicircle in the left half p-plane. By entering Figure 23 with values of α and β, the resulting poles of (12-7) may be read directly.

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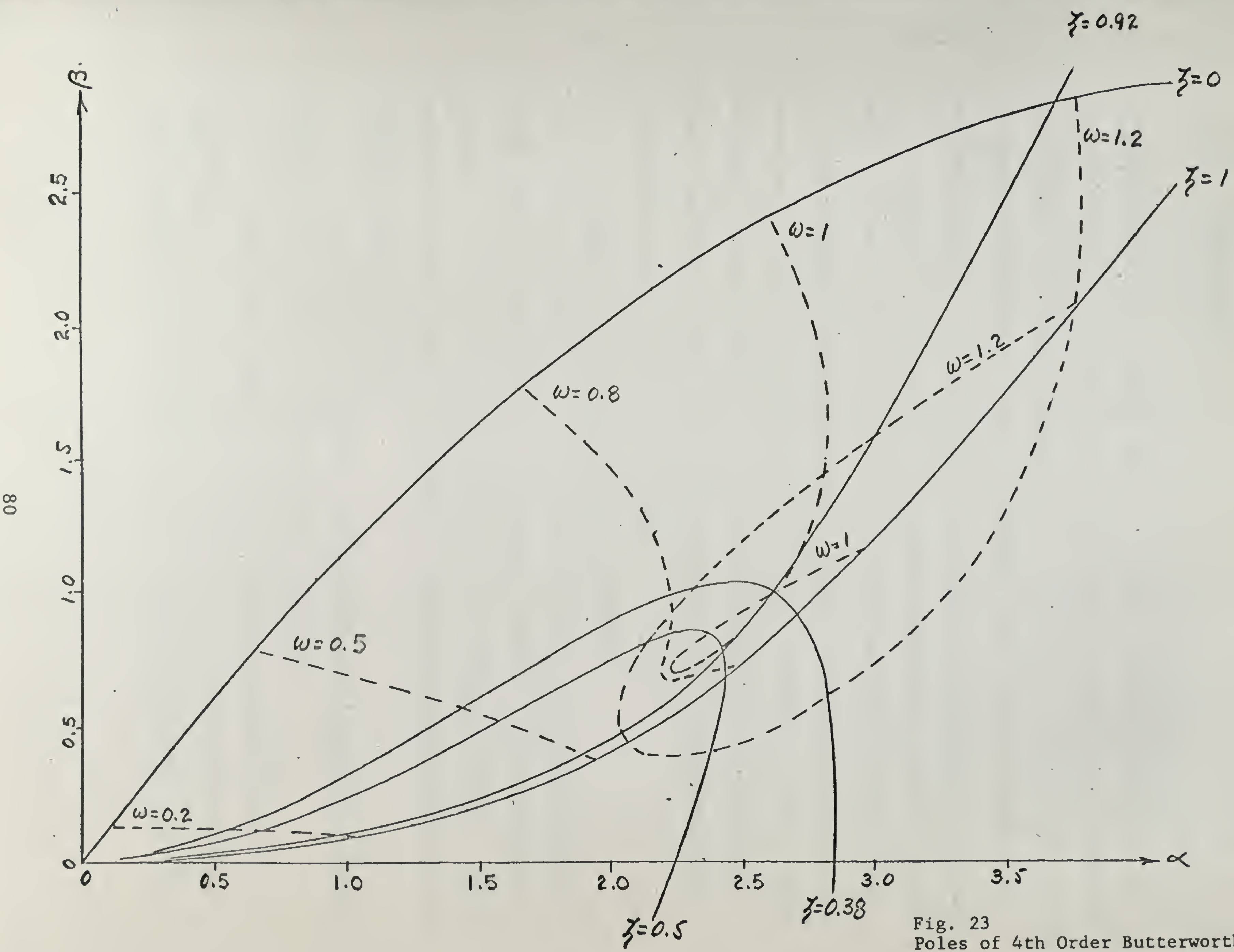

Fig. 23
Poles of 4th Order Butterworth Filter

13. Algebraic Design of Electric Networks by Parameter Plane Techniques

a. Introduction

Parameter values which will guarantee that certain poles and zeros of a rational network function appear at desired s-plane locations and at the same time guarantee that other specifications are met may be algebraically determined by employing parameter plane techniques. This may be accomplished without the necessity of drawing parameter plane curves and thus, more than two adjustable parameters may be simultaneously considered. In general it is necessary to have at least n adjustable parameters in order to make n specifications. While the design technique does not mathematically demand it, some preliminary investigation is desirable to determine whether or not a physically realizable set of parameter values is likely to be found which will meet the specifications. If such an investigation is not performed then it may be the case that the design technique will generate only sets of parameter values which are unrealizable.

b. Types of Specifications Which May be Used

Any specification on the network function or on the network behavior which can be expressed as an algebraic equation in the adjustable parameters can be used in conjunction with this technique of design. These specifications can include both pole-zero locations and performance specifications. Some of these are:

(a) Pole-zero locations. The parameter plane equations of Section 4 can be used to specify desired pole-zero locations.

(b) Zeta of a pair of complex roots may be specified without specifying the corresponding omega, or vice versa, by using the non-parametric equations of the constant zeta and omega loci presented in Section 5.

(c) Bandwidth. The equations of the constant bandwidth curves developed in Section 6 can be used to specify bandwidth.

(d) Q. The equations of the constant Q loci, Section 7, may be used to specify Q for second order systems.

(e) Maxima and minima in the frequency response can be specified by equating the derivative of the magnitude function to zero at the frequency where the maximum or minimum is desired.

(f) Infinite frequency or DC gain of a network may be specified by expressing these quantities as algebraic equations in the adjustable parameters.

c. <u>The Technique of Algebraic Design</u>

(1) Pole Specifications

Consider either the numerator or denominator of a rational network function written as the polynomial equation:

$$\sum_{k=0}^{n} a_k s^k = 0 \tag{13-1}$$

where the coefficients a_k are functions of the adjustable parameters q_i:

$$a_k = f_k(q_1, q_2, \ldots, q_i) \tag{13-2}$$

Parameter plane theory provides a number of algebraic equations which relate the root factors ζ, ω, σ to the parameter values q_i. These are:

(a) The basic parameter plane equations for real and complex roots:

1. For real roots:

$$\sum_{k=0}^{n} a_k \sigma^k = 0 \tag{13-3}$$

2. For complex roots:

$$\sum_{k=0}^{n} (-1)^k a_k \omega^k U_{k-j}(\zeta) = 0 \tag{13-4}$$

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where j is any integer.

(b) The constant zeta and omega loci in non-parametric form as developed in Section 5:

$$Z(a_k, \zeta) = 0 \tag{13-5}$$

$$W(a_k, \omega) = 0 \tag{13-6}$$

(2) <u>Other Specifications</u>

Consider the rational network function (13-7) which may be the transfer function of some network:

$$F(s)= \frac{\sum_{k=0}^{m} a_k s^k}{\sum_{k=0}^{n} b_k s^k} \tag{13-7}$$

where the coefficients a_k and b_k are different functions of the adjustable parameters q_i:

$$a_k = f_k(q_1, q_2, \ldots, q_i) \tag{13-8}$$

$$b_k = g_k(q_1, q_2, \ldots q_i) \tag{13-9}$$

Specifications on the behavior of F(s) must be written as algebraic equations in the parameters q_i in order to apply this design technique. This is straightforward for the common types of specifications. Sections 6 and 7 show how this is accomplished for specifications of Q and bandwidth.

(3) <u>Solution for the Parameter Values which Meet Specifications</u>

If r poles and zeros of the network function are specified, then Equations (13-3) through (13-6) may be employed as necessary to generate r independent equations in the adjustable parameters q_i. These r equations taken together with perhaps s other equations arising from other

specifications form a set of (s+r) equations which must simultaneously be satisfied. It is presumed that the number of adjustable parameters q_i is equal to or exceeds the number (s+r) of specifications. In order to obtain discrete solutions for the parameter values, the number of specifications must be made equal to the number of adjustable parameters. This may be achieved by fixing as many of the adjustable parameters as required to obtain this equality. Once the number of adjustable parameters equals the number (s+r) of specifications, the set of (s+r) algebraic equations may be simultaneously solved. Simple cases may be solved by hand, however the equations are generally complicated and require computer solution. Several iterative procedures are available to achieve this, and are usually some type of search routine which seeks parameter values which will force all equations to zero simultaneously.

In general the set of (s+r) equations is non-linear and its solution will result in several sets of parameter values which mathematically satisfy all equations simultaneously. Some of these solutions will be rejected because they are unrealizable. It should be kept in mind that the mathematical solution may not always be physically acceptable even though realizable. This situation may occur, for example, when specifying a maximum in the frequency response characteristic by selecting parameter values which will cause the derivative of the magnitude function to be zero at some specified frequency. The mathematics may indicate a solution which results in a minimum rather than the desired maximum. In any event solutions should be checked to ensure that they meet all of the specifications. If a solution exists which is both realizable and physically meets the specifications, then it will be found among the solutions to the (s+r) equations.

d. Example

Consider the case of the symmetrical parallel tee network which was investigated in Section 9. The zeros of the transfer function of this network are given by the roots of:

$$\alpha\beta\ p^3 + 2\ \beta\ p^2 + 2\ \beta\ p + 1 = 0$$

Since this equation has two variable parameters, two of its three roots may be specified. A preliminary analysis has indicated that complex roots are possible with real and positive parameter values. Assume that the parameter values are desired which will cause a pair of complex roots to have zeta equal to 0.2 and omega equal to 1.0. The basic parameter plane equations for complex roots are chosen with j = 0 and j = 1. These become:

$$0.84\ \alpha\beta\ -1.2\ \beta\ = 0$$

$$1.2\ \beta\ = 0.84$$

Solution of these equations produce the desired parameter values:

$$\alpha\ = 1.44$$

$$\beta\ = 0.70$$

By dividing the quadratic factor associated with the specified complex roots into the cubic equation (13-10), the real root is determined to be located at p = -1.

e. Example

Consider again the parallel-tee network of the previous example. Instead of completely specifying the complex zeros, let only the omega be specified, and instead of specifying zeta let the value of the real root be specified. Assume that the omega of the complex zeros is specified as 1.2 and that the real zero is to occur at s = -0.5. In order to obtain the

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desired parameter values which will result in these roots, the equations of the real roots and the constant omega loci are employed:

$$\sum_{k=0}^{n} a_k \sigma^k = 0$$

$$W(a_k, \omega) = 0$$

From Section 9, and with $\omega = 1.2$ and $\sigma = -0.5$, these equations are:

$$-0.125\alpha\beta + 0.50\beta - 1.0\beta + 1.0 = 0 \qquad (13\text{-}11)$$

$$2.99\,\alpha^2\beta^2 - 4.14\,\alpha\beta^2 + 2.88\beta - 1.0 = 0 \qquad (13\text{-}12)$$

Equation (13-11) is solved for α as a function of β :

$$\alpha = \frac{0.50\,\beta - 1.0}{-0.125\beta} \qquad (13\text{-}13)$$

Equation (13-13) is substituted into Equation (13-12) to form the quadratic in β:

$$64.33\beta^2 - 221.34\beta + 190.10 = 0 \qquad (13\text{-}14)$$

Equation (13-14) is solved for two values of β and then Equation (13-13) is used to obtain the corresponding values of α . The results are:

$\alpha_1 = 0.85$ $\qquad$ $\beta_1 = 1.65$

$\alpha_2 = 0.46$ $\qquad$ $\beta_2 = 1.79$

Upon checking of these results α_1 and β_1 are found to produce the desired zeros. α_2 and β_2 are rejected since they produce three real zeros, one of which has the desired sigma, rather than one real root and a pair of complex conjugate roots as desired. This situation was anticipated as discussed in the introduction to this section.

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f. Example

In the previous two examples only zeros of the transfer function were specified. In this example a pole is specified and a constraint is placed on the zeros. The symmetrical parallel tee network discussed in Section 9 is used as a vehicle. The transfer function of this network is:

$$\frac{E_o(p)}{E_i(p)} = \frac{\alpha\beta p^3 + 2\beta p^2 + 2\beta p + 1}{\alpha\beta p^3 + (2\beta + 2\alpha\beta + \alpha)p^2 + (2\beta + \alpha + 2)p + 1} \quad (13\text{-}15)$$

where p = RCs

Assume that a pole is required to be placed at p = -0.30 and that a pair of complex conjugate zeros is to have a damping ratio ζ = 0.4999. The equation of the constant zeta loci for the numerator polynomial which was obtained in Section 9 is used to express the constraint on zeta:

$$((8.00\alpha - 16.00)\beta + 8.00)\beta = 0 \quad (13\text{-}16)$$

The equation for the constant sigma curves for the denominator polynomial, also obtained in Section 9, is used to express the constraint on the pole:

$$0.153\alpha\beta - 0.420\beta - 0.050\alpha + 0.400 = 0 \quad (13\text{-}17)$$

Equation (13-16) is solved for α as a function of β :

$$\alpha = 2.00 - (1.00/\beta) \quad (13\text{-}18)$$

When Equation (13-18) is substituted into Equation (13-17) the following quadratic results:

$$\beta^2 + 1.52\beta - 1.84 = 0 \quad (13\text{-}19)$$

Equations (13-18) and (13-19) give the following pair of parameter values which mathematically satisfy the constraints placed on the pole and zeros of the transfer function:

$$\alpha_1 = 0.74 \qquad \beta_1 = 0.80$$

$$\alpha_2 = 2.43 \qquad \beta_2 = -2.31$$

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Upon checking, α_1 and β_1 are found to satisfy the specified constraints. α_2 and β_2 are rejected as unrealizable since β_2 is negative.

g. Example

Consider the loaded and null-adjusted symmetrical parallel tee network discussed. Presume that one wishes to select a value of the source resistance α, and the load resistance β, which will cause a pole to be located at $p = \sigma = -0.3$ and which will result in a value of $Q \doteq 0.34$. Referring to Section 10, Equation (10-2), it is seen that the specification on the real pole requires that:

$$-0.84\,\alpha\beta - 0.11\,\alpha - 0.11\,\beta + 1.40 = 0 \qquad (13\text{-}20)$$

Equation (10-10) of the same section gives the equation which must be satisfied to meet the specification on Q:

$$1.8496\,\alpha^2\beta^2 - 0.3008\,\alpha\beta^2 + 0.8496\,\beta^2 - 0.3008\,\alpha^2\beta - 4.4512\,\alpha\beta - 0.1504\,\beta + 0.8496\,\alpha^2 - 0.1504\,\alpha + 0.4624 = 0 \qquad (13\text{-}21)$$

Equations (13-20) and (13-21), when simultaneously solved by digital computer produce the solution:

$$\alpha = 1.95$$
$$\beta = 0.677$$

These values are physically realizable and meet the specifications.

h. Conclusions

The parameter plane technique presented in this section is the only design technique known to the author which allows parameter values to be determined which will cause both pole-zero and performance specifications to be simultaneously satisfied. The types of specifications which may be made are limited only by the requirement that they be expressible as algebraic equations in the adjustable parameters. For common specifications this is readily achieved.

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14. Conclusions and Recommendations for Further Work

a. Form of the Coefficients

The work which has been presented in Section 4 shows how parameter plane equations have been developed for polynomials with coefficients of the form:

$$a_k = b_k\alpha + c_k\beta + d_k\alpha\beta + f_k \tag{14-1}$$

This extension of the theory permits a wide class of network functions to be analyzed. There remains, however, cases where higher order terms in and appear in the coefficients. These include terms like: $\alpha^2\beta^2$, $\alpha^2\beta$, $\alpha\beta^2$, etc. Work currently being performed by Lt. A. R. Miller, USN at the U. S. Naval Postgraduate School has shown that it is possible to solve the parameter plane equations for coefficients of the form:

$$a_k = b_k\alpha^2 + c_k\beta^2 + d_k\alpha\beta + e_k\alpha + f_k\beta + g_k \tag{14-2}$$

Preliminary work done by Lt. R. E. Hudson, USN, also at the U. S. Naval Postgraduate School, has shown that it is theoretically possible to solve the parameter plane equations for coefficients of the form:

$$a_k = b_k\alpha^2\beta^2 + c_k\alpha^2\beta + d_k\alpha\beta^2 + e_k\alpha^2 + f_k\beta^2 + g_k\alpha\beta + h_k\alpha + q_k\beta + t_k \tag{14-3}$$

There remains however much to be done, particularily in interpreting the curves which result from solution of the parameter plane equations associated with (14-2) and (14-3).

It is, of course, desirable to allow even higher order combinations of α and β to appear in the polynomial coefficients than allowed by (14-3). At the current state of development it is not clear just how to achieve this.

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b. Non-parametric Equations of the Constant Zeta and Omega Loci

The derivation of the non-parametric equations of the constant zeta and omega loci presented in Section 5 permits these loci to be more readily constructed than was previously possible. More important, however, it forms the bridge between parameter plane theory and the theory of plane algebraic curves. To date this relationship has not been seriously investigated. It is the author's opinion that such investigation will lead to new and valuable information about the behavior of the loci. Such information may well lead to the development of sketching techniques which would considerably simplify the construction of the loci. Additionally it might permit new insight into the behavior of the zeta equals zero locus which is the stability limit for the poles of active networks.

c. Constant Bandwidth Curves

Simultaneous correlation between frequency response characteristics and root locations upon the parameter plane is now possible as a result of the development of the constant bandwidth curves which were developed in Section 6. Families of such curves permit the frequency response of a network to be sketched for any choice of parameter values within the range for which the curves are drawn.

The equations developed in Section 6 apply to coefficients in the transfer function of the form (14-1). When parameter plane techniques are developed for coefficients of the form (14-3) it will be necessary to extend the work of Section 6 to include this coefficient form. Preliminary work done by the author has shown that this is possible by following the same procedure as employed in Section 6.

d. Constant Q Loci

As defined by Morris (4), Q is a mathematical parameter of a second order characteristic equation. Thus defined, Q may be employed to characterize RC networks for which the conventional definitions fail to provide useable correlation between the network characteristics and the value of Q.

Loci of constant Q, developed in Section 7, permit the value of Q associated with any choice of network parameters to be read directly from the parameter plane. Alternately one may select parameter values from the parameter plane which will achieve a desired value of Q within the capabilities of the network.

e. Algebraic Design by Parameter Plane Methods

The algebraic design techniques presented in Section 13 permit m network parameter values to be determined which will guarantee that m specifications upon the network are met. These specifications can be either pole-zero locations or other specifications which can be written as algebraic equations in the m adjustable parameters, or both. As pointed out in Section 13, it is necessary to have the same number of adjustable parameters as specifications in order to obtain discrete sets of parameter values as solutions to the problem. Unless these specifications are realistic ones in terms of the network capabilities, then it may be the case that no physically realizable solutions will be produced by the design technique. It is more likely that a realizable solution would be found if the system had more adjustable parameters than specifications and if the specifications were stated as inequalities rather than strict equalities. For example, one might specify Q greater than 0.34 rather than strictly equal to 0.34 providing that this is acceptable. This procedure would lead to a set of equations

which would contain some strict equalities and other inequalities and hence other methods than those of Section 13 would have to be applied in order to obtain parameter values which would meet the specifications. The author suggests that non-linear programming techniques be investigated in connection with the solution of such a set of equations.

f. Summary

The basic parameter plane techniques, and the extensions to them presented here, provide a powerful tool for the analysis and design of variable parameter, active and passive, electric networks. These techniques are capable of providing information about the poles and zeros and sensitivity of variable parameter networks which is not possible by any other method currently known. It is, for two parameter problems, all and more that Evan's root locus technique is for single parameter problems. Like the root locus technique it is applicable to a much wider sphere of problems than that of electric networks. Subject to the restrictions on the form of the coefficients previously discussed, parameter plane techniques may be used to investigate the singularities of any variable coefficient rational function in the complex variable s.

15. Acknowledgements

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APPENDIX I

Inductive proof that:

$$U_i(\zeta)U_{j-1}(\zeta) - U_{i-1}(\zeta)U_j(\zeta) = -U_{i-j}(\zeta) \qquad \text{(I-1)}$$

for all integers i and j.

The inductive proof that (I-1) is true consists of the following steps:

1. Show that (I-1) is true for i = 2 and i = 3.
2. Assume that (I-1) is true for i = m-1 and i = m.
3. Show that (I-1) is true for i = m + 1, given that it is true

for i = m-1 and i = m-2.

Step 1a:

i = 2

$$U_2(\zeta)U_{j-1}(\zeta) - U_1(\zeta)U_j(\zeta) = -U_{2-j}(\zeta) \qquad \text{(I-2)}$$

but

$U_2(\zeta) = 2\zeta$

$U_1(\zeta) = 1$

Therefore (I-2) may be written:

$$2\zeta U_{j-1}(\zeta) - U_j(\zeta) = -U_{2-j}(\zeta) \qquad \text{(I-3)}$$

but $U_{-k}(\zeta) = -U_k(\zeta)$

Hence (I-3) becomes:

$$2\zeta U_{j-1}(\zeta) - U_j(\zeta) = U_{j-2}(\zeta) \qquad \text{(I-4)}$$

(I-4) is recognized as the recursion relation which defines the Chebyshev functions. Thus (I-2) is true for all j.

Step 1b:

i = 3

$$U_3(\zeta)\, U_{j-1}(\zeta) - U_2(\zeta)U_j(\zeta) = -U_{3-j}(\zeta) \qquad \text{(I-5)}$$

but:

$$U_3(\zeta) = 4\zeta^2-1$$

Therefore (I-5) may be written:

$$(4\zeta^2-1)U_{j-1}(\zeta) - 2\zeta U_j(\zeta) = U_{j-3}(\zeta) \tag{I-6}$$

Making use of the recursion relation for the Chebyshev functions, (I-4), it is possible to write (I-6);

$$(4\zeta^2-1)U_{j-1}(\zeta) - 4\zeta^2 U_{j-1}(\zeta) + 2\zeta U_{j-2}(\zeta) = U_{j-3}(\zeta) \tag{I-7}$$

Expanding (I-7) and cancelling where possible, produces the recursion relation for the Chebyshev functions, (I-4).

Therefore (I-5) is true.

Step 2:

Assume that the following equations are ture:

$$U_{m-1}(\zeta)U_{j-1}(\zeta) - U_{m-2}(\zeta)\, U_j(\zeta) = -U_{m-j-1}(\zeta) \tag{I-8}$$

$$U_m(\zeta)\, U_{j-1}(\zeta) - U_{m-1}(\zeta)U_j(\zeta) = -U_{m-j}(\zeta) \tag{I-9}$$

Step 3:

$$U_{m-j+1}(\zeta) = U_{m-j+1}(\zeta) \tag{I-10}$$

Applying the recursion relation for the Chebyshev functions, (I-4), to the left hand side of (I-10) and rewriting, one obtains:

$$2\zeta[U_{m-j}(\zeta)] - [U_{m-j-1}(\zeta)] = U_{m-j+1}(\zeta) \tag{I-11}$$

Substituting (I-8) and (I-9) into (I-11) one obtains:

$$2\zeta[U_m(\zeta)U_{j-1}(\zeta)-U_{m-1}(\zeta)\, U_j(\zeta)] - [U_{m-1}(\zeta)U_{j-1}(\zeta)-U_{m-2}(\zeta)U_j(\zeta)] = -U_{m-j+1}(\zeta) \tag{I-12}$$

Equation (I-12) may be written:

[illegible]

$$[2\zeta U_m(\zeta)-U_{m-1}(\zeta)]U_{j-1}(\zeta) - [2\zeta U_{m-1}(\zeta)-U_{m-2}(\zeta)]\, U_j(\zeta) = -U_{m-j+1}(\zeta) \tag{I-13}$$

By use of the recursion relation for the Chebyshev functions, (I-4), Equation (I-13) may be written:

$$U_{m+1}(\zeta)U_{j-1}(\zeta) - U_m(\zeta)U_j(\zeta) = -U_{m-j+1}(\zeta) \tag{I-14}$$

Thus (I-1) is true for i = m+1, given that it is true for i = m-1 and i = m-2. This concludes the proof that (I-1) is true for all i and j integer.

APPENDIX II

Proof that:

$$\sum_{k=0}^{n} (-1)^k a_k \omega^k U_{k-j}(\zeta) = 0 \qquad \text{(II-1)}$$

for all integers j and k

<u>Theorem</u>:

If: $$\sum_{k=0}^{n} (-1)^k a_k \omega^k U_k(\zeta) = 0 \qquad \text{(II-2)}$$

and if:

$$\sum_{k=0}^{n} (-1)^k a_k \omega^k U_{k-1}(\zeta) = 0 \qquad \text{(II-3)}$$

Then:

$$\sum_{k=0}^{n} (-1)^k a_k \omega^k U_{k-j}(\zeta) = 0 \quad \text{for all integers j and k} \qquad \text{(II-4)}$$

<u>Proof</u>:

Subtract (II-2) multiplied by $U_{j-1}(\zeta)$ from (II-3) multiplied by $U_j(\zeta)$. This gives:

$$\sum_{k=0}^{n} (-1)^k a_k \omega^k \left[U_k(\zeta)U_{j-1}(\zeta) - U_{k-1}(\zeta)U_j(\zeta)\right] = 0 \qquad \text{(II-5)}$$

Employing the results obtained in Appendix I, namely (I-1), Equation (II-5) may be written:

$$\sum_{k=0}^{n} (-1)^k a_k \omega^k \left[-U_{k-j}(\zeta)\right] = 0 \qquad \text{(II-6)}$$

Multiplying (II-6) by -1 concludes the proof of (II-4).

[illegible]

APPENDIX III

Table of the Chebyshev Functions

The Chebyshev Functions $T_k(\zeta)$ and $U_k(\zeta)$ are defined by the following recursion relations:

$$T_{k+1}(\zeta) - 2\zeta T_k(\zeta) + T_{k-1}(\zeta) = 0$$

$$T_0(\zeta) = 1$$

$$T_1(\zeta) =$$

$$U_{k+1}(\zeta) - 2\zeta U_k(\zeta) + U_{k-1}(\zeta) = 0$$

$$U_0(\zeta) = 0$$

$$U_1(\zeta) = 1$$

The first few of these functions are tabulated below:

k	$T_k(\zeta)$	$U_k(\zeta)$
0	1	0
1	ζ	1
2	$2\zeta^2-1$	2ζ
3	$4\zeta^3-3\zeta$	$4\zeta^2-1$
4	$8\zeta^4-8\zeta^2+1$	$8\zeta^3-4\zeta$